1. Let $X$ be a continuous random variable with density function $f(x) = \begin{cases} cx^3 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

Find the value of $c$ that makes $f(x)$ a density function.

The property of density functions that we will use to determine $c$ is that $\int_{-\infty}^{\infty} f(x) \, dx = 1$.

Here, $\int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{1} f(x) \, dx = \int_{0}^{1} cx^3 \, dx = \left[ \frac{cx^4}{4} \right]_{x=0}^{x=1} = \frac{c}{4} - 0 = 1$.

Since $\frac{c}{4} = 1 \implies c = 4$, and $f(x) = \begin{cases} 4x^3 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

2. Given the density function of the previous problem, compute $P\left(\frac{1}{4} \leq X \leq \frac{3}{4}\right)$.

Given a density function for a continuous random variable $X$, $P(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx$.

Therefore, $P\left(\frac{1}{4} \leq X \leq \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} f(x) \, dx = \int_{\frac{1}{4}}^{\frac{3}{4}} 4x^3 \, dx = \left[ x^4 \right]_{x=\frac{1}{4}}^{x=\frac{3}{4}} = \left(\frac{3}{4}\right)^4 - \left(\frac{1}{4}\right)^4 = \frac{5}{16}$.

3. Given that the continuous random variable $X$ has the distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \sin \left(\frac{\pi}{2} x\right) & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases},$$

find the density function $f(x)$.

The distribution function $F(x)$ and the density function $f(x)$ of a continuous random variable are related as follows: $F(x) = \int_{-\infty}^{x} f(t) \, dt$ and $f(x) = \frac{d}{dx} [F(x)] = F'(x)$.

Therefore, $f(x) = F'(x) = \frac{\pi}{2} \cos \left(\frac{\pi}{2} x\right)$.

4. The continuous random variable $X$ has the density function $f(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

Find the mean and the variance.

For a continuous random variable, the mean $\mu$ is given by $\mu = E[X] = \int_{-\infty}^{\infty} x f(x) \, dx$.

So $\mu = E[X] = \int_{0}^{1} x^3 \, dx = \left[ \frac{3x^4}{4} \right]_{x=0}^{x=1} = \frac{3}{4}$.

For the variance, we must first find $E[X^2]$.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) \, dx = \int_{0}^{1} x^2 \, dx = \left[ \frac{3x^3}{3} \right]_{x=0}^{x=1} = \frac{3}{3} = 1.$$

Therefore, $V(X) = E[X^2] - \mu^2 = \frac{3}{3} - \left(\frac{3}{4}\right)^2 = 0.375$.

5. If a parachutist lands at a **random** point on a line between two markers $A$ and $B$, find:
(a) The probability that he is closer to $A$ than to $B$.

Since the probability of the parachutist landing at any point in the interval $[A, B]$ is the same as the probability of the parachutist landing at any other point in the interval, the distribution is Uniform (or rectangular) and the density function is of the form

$$f(x) = \begin{cases} \frac{1}{B-A} & \text{for } A \leq X \leq B \\ 0 & \text{otherwise} \end{cases}$$

Observe: The probability that the parachutist is closer to $A$ than to $B$ is $P(A \leq X \leq \frac{A+B}{2})$, since $\frac{A+B}{2}$ is the midpoint of the interval $[A, B]$. Therefore,

$$P(A \leq X \leq \frac{A+B}{2}) = F\left(\frac{A+B}{2}\right) = \int_A^{\frac{A+B}{2}} \frac{1}{B-A} \, dx = \left[\frac{x}{B-A}\right]_{A}^{\frac{A+B}{2}} = \frac{A+B}{2(B-A)} - \frac{A}{B-A} = \frac{B-A}{2(B-A)} = \frac{1}{2}.$$

(b) The probability that his distance to $A$ is more than twice his distance to $B$.

To find this, we divide the interval $[A, B]$ into three subintervals of equal length, namely:

$$[A, \frac{2A+B}{3}], \quad \left[\frac{2A+B}{3}, \frac{A+2B}{3}\right], \quad \left[\frac{A+2B}{3}, B\right].$$

Note that if the parachutist’s distance to $A$ is more than twice his distance to $B$, then he must land in the interval $\left[\frac{A+2B}{3}, B\right]$. Therefore, the probability that his distance to $A$ is more than twice his distance to $B$ is the same as $P\left(\frac{A+2B}{3} \leq X \leq B\right) = \int_{\frac{A+2B}{3}}^{B} \frac{1}{B-A} \, dx = \left[\frac{x}{B-A}\right]_{\frac{A+2B}{3}}^{B} = \frac{B-A}{3(B-A)} = \frac{1}{3}$.

6. The change in depth of a river from one day to the next, measured (in feet) at a specific location, is a random variable $X$ with the following density function:

$$f(x) = \begin{cases} k & \text{for } -2 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

(a) determine the value of $k$.

To find $k$ we realize that $\int_{-\infty}^{\infty} f(x) \, dx = 1 \implies \int_{-2}^{2} k \, dx = [kx]^2_{-2} = 4k = 1 \\
\implies k = \frac{1}{4}$

(b) Find $P(-.3 \leq X \leq .5)$

$$P(-.3 \leq X \leq .5) = \int_{-3}^{5} \frac{1}{4} \, dx = \left[\frac{1}{4}x\right]_{-3}^{5} = \frac{1}{4}(5) - \frac{1}{4}(-3) = .2$$

7. The weekly amount spent for maintenance and repairs for a certain company was observed, over a long period of time, to be approximately normally distributed with a mean of $400 and a standard deviation of $20. If $450 is budgeted for next week, what is the probability that the actual costs will exceed the budgeted amount?

We want: $P(X \geq 450)$. We must “standardize” this random variable before we can use the table in the back of the book. $P(X \geq 450) = P(X - \mu \geq 450 - 400) = \ldots$
10. A fair die is tossed

8. The grade point averages of a large number of college students are approximately normally
distributed with a mean of 2.4 and a standard deviation of 0.8. What percentage of students
will possess a grade point average in excess of 3.0?

We want: $P(X \geq 3.0)$. We must “standardize” this random variable before we can use the
table in the back of the book. $P(X \geq 3.0) = P(X - \mu \geq 3.0 - 2.4) = P(X - \mu \geq 0.6) =
\frac{X - \mu}{\sigma} \geq \frac{0.6}{0.8} = P(Z \geq 0.75) = .5 - P(0 \leq Z \leq 0.75) = .5 - .2734 = .2266$

9. A machine is shut down for repairs if a random sample of 100 items selected from the daily
output of the machine reveals at least 15% defectives. Assume that the daily output is a
large number of items.) If the machine is, in fact, producing only 10% defective items, find
the probability that it will be shut down on a given day. (If you use the table on p. 800,
you may have to interpolate.)

Since “the daily output is a large number of items”, we can assume that the distribution is
Binomial. (i.e. sample size is small compared to the population size.) So the probability
of getting a defective is $p = 0.1$. Given a random sample of size $n = 100$, we want the
probability that proportion of defectives is greater than 0.15 (i.e. we want $P\left(\frac{X}{n} \geq 0.15\right)$).

Here, $np = 10 \geq 5$ and $p = .1 \leq .5$, so we are justified in using the Normal distribution to
approximate the Binomial. If $\frac{X}{n}$ is our random variable, then the mean is $p = 0.1$ and the
standard deviation is $\sqrt{\frac{npq}{n}} = 0.03$.

Again, we want: $P\left(\frac{X}{n} \geq 0.15\right)$. First, we must “standardize” our random variable.

$P\left(\frac{X}{n} \geq 0.15\right) = P\left(\frac{X}{n} - p \geq 0.15 - 0.1\right) = P\left(\frac{X}{n} - p \geq 0.05\right) = P\left(\frac{X - np}{\sqrt{npq}} \geq \frac{0.05}{0.03}\right) = P(Z \geq 1.66666) =
.5 - P(0 \leq Z \leq 1.66666) = .5 - .4522 = 0.0478$

Note: Since I wanted $Z \leq 1.66666$, I interpolated between 1.6 and 1.7 in Table 1, adding
0.0007 to .4515 to get .4522.

10. A fair die is tossed 300 times. What is the probability of “getting a six” at least 70 times?
(Since $n$ is large, do not use the shifting technique.)

First of all, this is a binomial distribution, because the trials are independent (our chances of
getting a six on a toss are not affected by what we get on any other toss) and we have only
successes (six) or failures (not a six) as possible outcomes. Also, we won’t use the “shifting”
trick on this problem - the number of trials and successes is large enough that it shouldn’t
make a much of a difference. Here, $X$ is the number of sixes that we get. The mean will be
$np = 300 \cdot \frac{1}{6} = 50$, and the standard deviation will be $\sqrt{npq} = \sqrt{300 \cdot \frac{1}{6} \cdot \frac{5}{6}} = 6.455$.

We want: $P(X \geq 70) = P(X - np \geq 70 - 50) = P(X - np \geq 20) = P\left(\frac{X - np}{\sqrt{npq}} \geq \frac{20}{6.455}\right) =
P(Z \geq 3.098) = .5 - P(0 \leq Z \leq 3.098) = .5 - .499 = .001$
11. The lifetime (in hours) $X$ of a certain electronic component is a random variable with density function

$$f(x) = \begin{cases} \frac{1}{100}e^{-\frac{x}{100}} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Three of these components operate independently in a piece of equipment. The equipment fails if at least two of the components fail. Find the probability that the equipment operates for at least 200 hours without failure.

If we restrict our attention to a single component, the probability that the component lasts for at least 200 hours is

$$\int_{200}^{\infty} \frac{1}{100}e^{-\frac{x}{100}} \, dx = \lim_{b \to \infty} \int_{200}^{b} \frac{1}{100}e^{-\frac{x}{100}} \, dx = \lim_{b \to \infty} \left[-e^{-\frac{x}{100}}\right]_{200}^{b} = \lim_{b \to \infty} e^{-\frac{b}{100}} - e^{-\frac{200}{100}} = e^{-2} = .13534$$

Let $E_1$ be the event that the first component lasts more than 200 hours.

Let $E_2$ be the event that the second component lasts more than 200 hours.

Let $E_3$ be the event that the third component lasts more than 200 hours.

We want $P$ (equipment lasts for more than 200 hours) =

$P$ (two or more components last more than 200 hours) =

$$P[(E_1 \text{ and } E_2 \text{ and } E_3) \text{ or } (E_1^c \text{ and } E_2 \text{ and } E_3) \text{ or } (E_1 \text{ and } E_2^c \text{ and } E_3) \text{ or } (E_1 \text{ and } E_2 \text{ and } E_3^c)] = P(E_1 \text{ and } E_2 \text{ and } E_3) + P(E_1^c \text{ and } E_2 \text{ and } E_3) + P(E_1 \text{ and } E_2^c \text{ and } E_3) + P(E_1 \text{ and } E_2 \text{ and } E_3^c) =$$

(Because the events are mutually exclusive.)

$$= P(E_1)P(E_2)P(E_3) + P(E_1^c)P(E_2)P(E_3) + P(E_1)P(E_2^c)P(E_3) + P(E_1)P(E_2)P(E_3^c) =$$

(Because the events are independent.)

$$= (.13534)(.13534)(.13534) + (.86466)(.13534)(.13534) + (.13534)(.86466)(.13534) + (.13534)(.13534)(.86466) = .049993$$