MTH 1125 - Test 2 (2pm Class) - Solutions

FALL 2023

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\frac{d}{dx}\left[2x^6+3x^4+8x^3+12x^2+20x+40\sqrt{x}+10\right]=$

$$\frac{d}{dx}\left[2x^6+3x^4+8x^3+12x^2+20x+40x^{\frac{1}{2}}+10\right]$$

$$=2\left[6x^5\right]+3\left[4x^3\right]+8\left[3x^2\right]+12\left[2x\right]+20+40\left[\tfrac{1}{2}x^{-\frac{1}{2}}\right]+0$$

$$=12x^5+12x^3+24x^2+24x+20+20x^{-\frac{1}{2}}$$

i.e., $\frac{d}{dx}\left[2x^6+3x^4+8x^3+12x^2+20x+40\sqrt{x}+10\right]=12x^5+12x^3+24x^2+24x+20+20x^{-\frac{1}{2}}$

2. Compute: $\frac{d}{dx}\left[\left(\sin(x)+8x^5\right)\left(\cos(x)+4x+2\right)\right]=$

$$\frac{d}{dx}\left[\underbrace{\left(\sin(x)+8x^5\right)}_{1^{st}}\underbrace{\left(\cos(x)+4x+2\right)}_{2^{nd}}\right]=\underbrace{\left(\cos(x)+40x^4\right)}_{1^{\text{st}}\text{ prime}}\cdot\underbrace{\left(\cos(x)+4x+2\right)}_{2^{nd}}+\underbrace{\left(-\sin(x)+4\right)}_{2^{\text{nd}}\text{ prime}}\cdot\underbrace{\left(\sin(x)+8x^5\right)}_{1^{st}}$$

$\frac{d}{dx}\left[\left(\sin(x)+8x^5\right)\left(\cos(x)+4x+2\right)\right]=\left(\cos(x)+40x^4\right)\left(\cos(x)+4x+2\right)+\left(-\sin(x)+4\right)\left(\sin(x)+8x^5\right)$

3. Compute: $\frac{d}{dx}\left[\frac{3x^6+6x^3+18x}{3x^3+9x+9}\right]=$

$$\frac{d}{dx}\left[\frac{\overbrace{3x^6+6x^3+18x}^{\text{top}}}{\underbrace{3x^3+9x+9}_{\text{Bottom}}}\right]=\frac{\overbrace{\left(18x^5+18x^2+18\right)}^{\text{top prime}}\cdot\overbrace{\left(3x^3+9x+9\right)}^{\text{bottom}}-\overbrace{\left(9x^2+9\right)}^{\text{bottom prime}}\cdot\overbrace{\left(3x^6+6x^3+18x\right)}^{\text{top}}}{\underbrace{\left(3x^3+9x+9\right)^2}_{\text{bottom squared}}}$$

i.e., $\frac{d}{dx}\left[\frac{3x^6+6x^3+18x}{3x^3+9x+9}\right]=\frac{\left(18x^5+18x^2+18\right)\left(3x^3+9x+9\right)-\left(9x^2+9\right)\left(3x^6+6x^3+18x\right)}{\left(3x^3+9x+9\right)^2}$

4. Compute: $\frac{d}{dx}\left[\left(5x^3+6x^2+9x\right)^{15}\right] =$ This is the derivative of a function raised to a power.

$$\frac{d}{dx}\left[\left(5x^3+6x^2+9x\right)^{15}\right] = \underbrace{15\left(5x^3+6x^2+9x\right)^{14}}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(15x^2+12x+9\right)}_{\substack{\text{derivative}\\ \text{of inner}}}$$

> i.e., $\frac{d}{dx}\left[\left(5x^3+6x^2+9x\right)^{15}\right] = 15\left(5x^3+6x^2+9x\right)^{14}\left(15x^2+12x+9\right)$

5. Given that $f(x) = -2x^2+3x+3$, give the *equation* of the line tangent to the graph of $f(x)$ at the point $(2,1)$.

   We need two things:

   i. A **point** on the line (We have that: $(x_1, y_1) = (2,1)$)

   ii. The **slope** of the line (This is $f'(x_1)$)

   $f'(x) = -4x + 3$

   At the point $(x_1, y_1) = (2,1)$, **the slope is** $f'(2) = -4(2) + 3 = -5$

   We will use the Point-Slope equation of a line:

   $(y - y_1) = m(x - x_1)$ (Where m is the slope and (x_1, y_1) is a known point on the line.)

   Thus, the equation of the line tangent to the graph of $f(x)$ is:

   $(y-1) = -5(x-2)$

> The equation of the line tangent is $(y-1) = -5(x-2)$

6. Given that $w = \tan(u)$ and that $u = 3t^2 + 3t + 3$; compute $\frac{dw}{dt}$ **using the Liebniz form of the Chain Rule.** (In particular, when doing this exercise, *write explicitly the Liebniz form of the chain rule that you are going to use.)*

We know:

$\frac{dw}{du} = \sec^2(u)$

$\frac{du}{dt} = 6t + 3$

We want: $\frac{dw}{dt}$

By the Liebniz form of the Chain Rule:

$$\frac{dw}{dt} = \frac{dw}{du}\frac{du}{dt} = \sec^2(u)(6t+3) = \underbrace{\sec^2\left(3t^2+3t+3\right)(6t+3)}_{\substack{\text{express solely in terms of} \\ \text{independent variable } u}}$$

i.e. $\frac{dw}{dt} = \sec^2(3t^2+3t+3)(6t+3)$

7. Compute: $\frac{d}{dx}[\sec(3x^4+6x^3)] =$

Outer:	=	$\sec(\quad)$
Deriv. of outer	=	$\sec(\quad)\tan(\quad)$

$$\frac{d}{dx}\left[\begin{array}{c} \sec\left(\underbrace{3x^4+6x^3}\right) \\ \uparrow \qquad\quad \uparrow \\ \text{outer} \quad \text{inner} \end{array}\right] = \underbrace{\sec\left(3x^4+6x^3\right)\tan\left(3x^4+6x^3\right)}_{\substack{\text{Deriv of outer,} \\ \text{eval. at inner}}} \cdot \underbrace{\left(12x^3+18x^2\right)}_{\substack{\text{deriv. of} \\ \text{inner}}}$$

i.e., $\frac{d}{dx}[\sec(3x^4+6x^3)] = \sec(3x^4+6x^3)\tan(3x^4+6x^3)(12x^3+18x^2)$

8. Compute: $\frac{d}{dx}\left[\left(\frac{4x^3+6x}{4x^4+8x^2+16}\right)^{10}\right] =$ In the broadest sense, this is the derivative of a *function raised to a power* - USE the CHAIN RULE.

$$\frac{d}{dx}\left[\underbrace{\left(\frac{4x^3+6x}{4x^4+8x^2+16}\right)^{10}}_{(g(x))^n}\right] = \underbrace{10\left(\frac{4x^3+6x}{4x^4+8x^2+16}\right)^{9}}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\frac{4x^3+6x}{4x^4+8x^2+16}\right]\right)}_{\substack{\text{deriv of}\\ \text{inner Function}}}$$

$$= 10\left(\frac{4x^3+6x}{4x^4+8x^2+16}\right)^{9} \underbrace{\frac{(12x^2+6)(4x^4+8x^2+16) - (16x^3+16x)(4x^3+6x)}{(4x^4+8x^2+16)^2}}_{\substack{\text{quotient}\\ \text{rule}}}$$

i.e., $\frac{d}{dx}\left[\left(\frac{4x^3+6x}{4x^4+8x^2+16}\right)^{10}\right] = 10\left(\frac{4x^3+6x}{4x^4+8x^2+16}\right)^{9} \frac{(12x^2+6)(4x^4+8x^2+16) - (16x^3+16x)(4x^3+6x)}{(4x^4+8x^2+16)^2}$

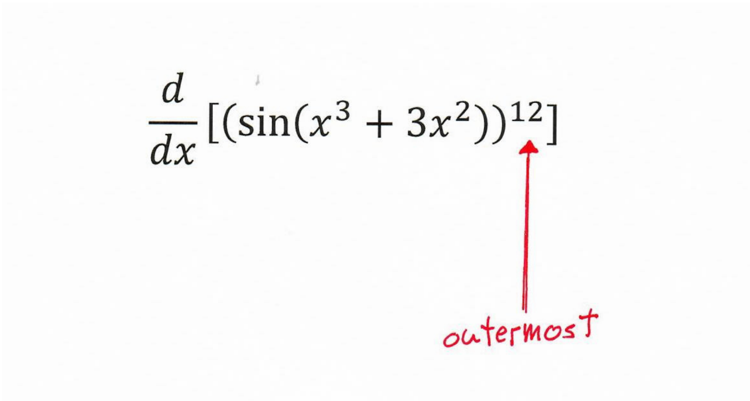
9. Compute: $\frac{d}{dx}\left[\sin^{12}\left(x^3+3x^2\right)\right] =$

Let's rewrite this:

$\frac{d}{dx}\left[\left(\sin\left(x^3+3x^2\right)\right)^{12}\right]$

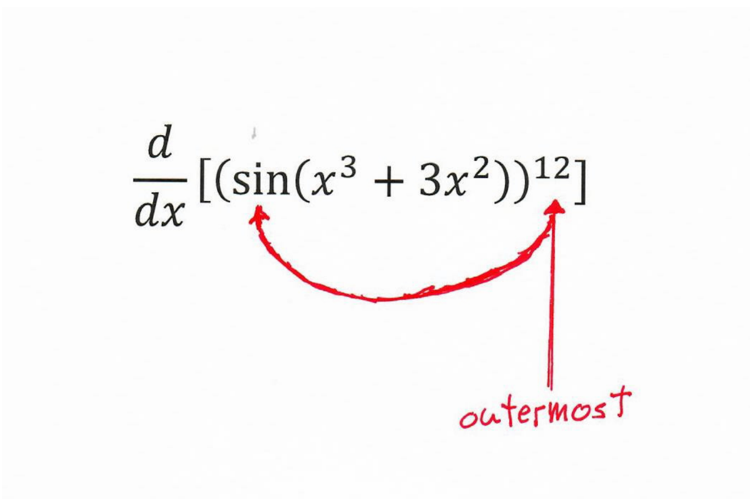
This is the composition of *three* functions.

Differentiate the outermost function and evaluate it at everything inside



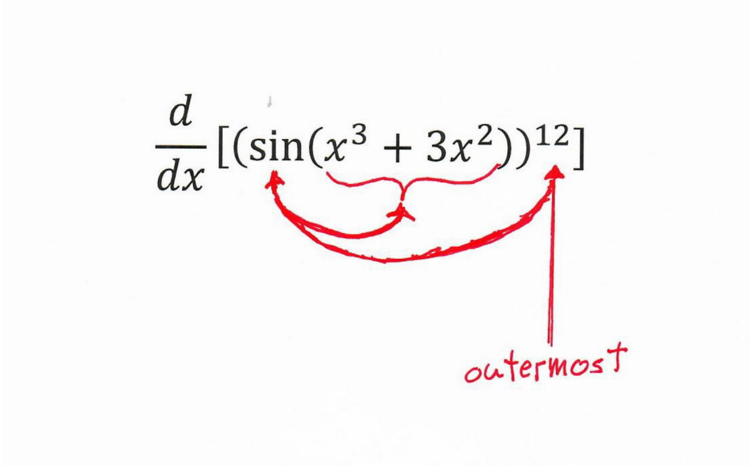
This yields: $12\left(\sin\left(x^3+3x^2\right)\right)^{11}$

Next: Multiply by the derivative of the next outermost function and evaluate it at everything inside of it.



This yields: $12\left(\sin\left(x^3+3x^2\right)\right)^{11}\cdot\cos\left(x^3+3x^2\right)$

Finally: Multiply by the derivative of the innermost function.



This yields: $12\left(\sin\left(x^3+3x^2\right)\right)^{11}\cdot\cos\left(x^3+3x^2\right)\cdot\left(3x^2+6x\right)$

i.e., $\frac{d}{dx}\left[\sin^{12}\left(x^3+3x^2\right)\right]=12\left(\sin\left(x^3+3x^2\right)\right)^{11}\cdot\cos\left(x^3+3x^2\right)\cdot\left(3x^2+6x\right)$

Alternatively:

Re-Write!

$\frac{d}{dx}\left[\sin^{12}\left(x^3+3x^2\right)\right]=\frac{d}{dx}\left[\left(\sin\left(x^3+3x^2\right)\right)^{12}\right]$

In the broadest sense, this is *the derivative of a function raised to a power*

$$\frac{d}{dx}\left[\left(\sin\left(x^3+3x^2\right)\right)^{12}\right]=\underbrace{12\left(\sin\left(x^3+3x^2\right)\right)^{11}}_{\substack{\text{power rule}\\ \text{as usual}}}\cdot\underbrace{\left(\frac{d}{dx}\left[\sin\left(x^3+3x^2\right)\right]\right)}_{\substack{\text{derivative}\\ \text{of inner}}}$$

$$=12\left(\sin\left(x^3+3x^2\right)\right)^{11}\cdot\underbrace{\left[\cos\left(x^3+3x^2\right)\cdot\left(3x^2+6x\right)\right]}_{\substack{\text{Chain}\\ \text{Rule}}}$$

i.e., $\frac{d}{dx}\left[\left(\sin\left(x^3+3x^2\right)\right)^{12}\right]=12\left(\sin\left(x^3+3x^2\right)\right)^{11}\cdot\cos\left(x^3+3x^2\right)\cdot\left(3x^2+6x\right)$

10. Given that $5x^4 - x^4y^4 = \tan(y)$, compute $\frac{dy}{dx}$

i. Differentiate both sides w.r.t. x

$$\frac{d}{dx}\left[5x^4 - \underbrace{x^4}_{1^{\text{st}}}\underbrace{y^4}_{2^{\text{nd}}}\right] = \frac{d}{dx}[\tan(y)]$$

$$\Rightarrow 20x^3 - \left(\underbrace{4x^3}_{1^{\text{st}}\text{ prime}} \cdot \underbrace{y^4}_{2^{\text{nd}}} + \underbrace{4y^3\frac{dy}{dx}}_{2^{\text{nd}}\text{ prime}} \cdot \underbrace{x^4}_{1^{\text{st}}}\right) = \sec^2(y) \cdot \frac{dy}{dx}$$

Simplifying, we have:

$$20x^3 - 4x^3y^4 - 4x^4y^3\frac{dy}{dx} = \sec^2(y)\frac{dy}{dx}$$

ii. Solve algebraically for $\frac{dy}{dx}$

a. Get $\frac{dy}{dx}$ terms on left side, all other terms on right side

$$\Rightarrow -4x^4y^3\frac{dy}{dx} - \sec^2(y)\frac{dy}{dx} = -20x^3 + 4x^3y^4$$

b. Factor out $\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx}\left(-4x^4y^3 - \sec^2(y)\right) = -20x^3 + 4x^3y^4$$

c. Divide both sides by the cofactor of $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-20x^3+4x^3y^4}{-4x^4y^3-\sec^2(y)} = \frac{20x^3-4x^3y^4}{4x^4y^3+\sec^2(y)}$$

$$\boxed{\frac{dy}{dx} = \frac{-20x^3+4x^3y^4}{-4x^4y^3-\sec^2(y)} = \frac{20x^3-4x^3y^4}{4x^4y^3+\sec^2(y)}}$$

11. Given that $f(x) = 6x^2 - 8x + 4$, compute $f'(x)$ **using the definition of derivative.** (i.e., using the "limit process.")

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[6(x+\Delta x)^2 - 8(x+\Delta x) + 4\right] - \left[6x^2 - 8x + 4\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[6\left(x^2 + 2x\Delta x + \Delta x^2\right) - 8(x+\Delta x) + 4\right] - \left[6x^2 - 8x + 4\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[6x^2 + 12x\Delta x + 6\Delta x^2 - 8x - 8\Delta x + 4\right] - \left[6x^2 - 8x + 4\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{12x\Delta x + 6\Delta x^2 - 8\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(12x + 6\Delta x - 8)}{\Delta x} = \lim_{\Delta x \to 0} (12x + 6\Delta x - 8) = 12x + 6(0) - 8 = 12x - 8$$

$$\boxed{\text{i.e., } f'(x) = 12x - 8}$$

Extra (Wow! 10 Points)

Given that $L'(x) = \frac{1}{\sqrt{x^2+1}}$ (i.e., $\frac{d}{dx}[L(x)] = \frac{1}{\sqrt{x^2+1}}$); compute $\frac{d}{dx}[L(\tan(x))]$

Outer:	$=$	$L(\quad)$
Deriv. of outer	$=$	$\frac{1}{\sqrt{(\quad)^2+1}}$

$$\frac{d}{dx}\Big[\underset{\uparrow \atop \text{outer}}{L}\big(\underbrace{\tan(x)}_{\uparrow \atop \text{inner}}\big)\Big] = \underbrace{\frac{1}{\sqrt{\left(\underbrace{\tan(x)}\right)^2 + 1}}}_{\text{Deriv of outer, eval. at inner}} \cdot \underbrace{\sec^2(x)}_{\text{deriv. of inner}} = \frac{\sec^2(x)}{\sqrt{(\tan(x))^2+1}} = \frac{\sec^2(x)}{\sqrt{\sec^2(x)}} = \frac{\sec^2(x)}{\sec(x)} =$$

$\sec(x)$

$$\boxed{\text{i.e., } \frac{d}{dx}[L(\tan(x))] = \frac{\sec^2(x)}{\sqrt{(\tan(x))^2+1}} = \sec(x)}$$