MTH 1126 - Test #1 - 9 am Class

Spring 2023

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Show CLEARLY how you arrive at your answers

1. Compute: $\frac{d}{dx}\left[e^{\tan\left(x^2\right)}\right] =$

$$\underbrace{\frac{d}{dx}\left[e^{\tan\left(x^2\right)}\right]}_{\frac{d}{dx}[e^u]} = \underbrace{e^{\tan\left(x^2\right)}}_{e^u} \cdot \underbrace{\frac{d}{dx}\left[\tan\left(x^2\right)\right]}_{\frac{du}{dx}} = e^{\tan\left(x^2\right)} \cdot \sec^2\left(x^2\right) \cdot 2x$$

> i.e., $\frac{d}{dx}\left[e^{\tan\left(x^2\right)}\right] = e^{\tan\left(x^2\right)} \cdot \sec^2\left(x^2\right) \cdot 2x$
>
> Or: $\frac{d}{dx}\left[e^{\tan\left(x^2\right)}\right] = 2x\sec^2\left(x^2\right) e^{\tan\left(x^2\right)}$

2. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{e^x}{2x^3+x^2}}\right)\right] =$

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{e^x}{2x^3+x^2}}\right)\right] = \frac{d}{dx}\left[\ln\left(\left(\frac{e^x}{2x^3+x^2}\right)^{\frac{1}{2}}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{e^x}{2x^3+x^2}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(e^x\right) - \ln\left(2x^3+x^2\right)\right)\right]$$

$$= \frac{d}{dx}\left[\frac{1}{2}\left(x - \ln\left(2x^3+x^2\right)\right)\right] = \frac{1}{2}\left(1 - \underbrace{\frac{1}{2x^3+x^2}}_{\frac{1}{u}} \cdot \underbrace{\left(6x^2+2x\right)}_{\frac{du}{dx}}\right) = \frac{1}{2}\left(1 - \frac{6x+2}{2x^2+x}\right) = \frac{1}{2} - \frac{3x+1}{2x^2+x}$$

> i.e., $\frac{d}{dx}\left[\ln\sqrt{\frac{e^x}{2x^3+x^2}}\right] = \frac{1}{2} - \frac{3x+1}{2x^2+x}$

(Alternative Solution Appears on the Following Page)

Alternative Solution:

$$\frac{d}{dx}\left[\ln\left(\sqrt{\tfrac{e^x}{2x^3+x^2}}\right)\right] = \frac{d}{dx}\left[\underbrace{\ln\left[\left(\frac{e^x}{2x^3+x^2}\right)^{\frac{1}{2}}\right]}_{\ln(u)}\right] = \underbrace{\frac{1}{\left(\frac{e^x}{2x^3+x^2}\right)^{\frac{1}{2}}}}_{\frac{1}{u}} \cdot \underbrace{\left(\frac{d}{dx}\left[\left(\frac{e^x}{2x^3+x^2}\right)^{\frac{1}{2}}\right]\right)}_{\frac{du}{dx}}$$

$$= \left(\tfrac{e^x}{2x^3+x^2}\right)^{-\frac{1}{2}} \cdot \tfrac{1}{2}\left(\tfrac{e^x}{2x^3+x^2}\right)^{-\frac{1}{2}} \underbrace{\frac{e^x\left(2x^3+x^2\right) - \left(6x^2+2x\right)e^x}{\left(2x^3+x^2\right)^2}}_{\text{Quotient Rule}}$$

$$= \tfrac{1}{2}\left(\tfrac{e^x}{2x^3+x^2}\right)^{-\frac{1}{2}}\left(\tfrac{e^x}{2x^3+x^2}\right)^{-\frac{1}{2}} \tfrac{e^x\left(\left(2x^3+x^2\right)-\left(6x^2+2x\right)\right)}{\left(2x^3+x^2\right)^2}$$

$$= \tfrac{1}{2}\left(\tfrac{e^x}{2x^3+x^2}\right)^{-1} \tfrac{e^x\left(2x^3-5x^2-2x\right)}{\left(2x^3+x^2\right)^2} = \tfrac{1}{2}\left(\tfrac{2x^3+x^2}{e^x}\right)\tfrac{e^x\left(2x^3-5x^2-2x\right)}{\left(2x^3+x^2\right)^2} = \tfrac{1}{2}\tfrac{\left(2x^3-5x^2-2x\right)}{\left(2x^3+x^2\right)}$$

$$= \tfrac{1}{2}\tfrac{\left(2x^3+x^2-x^2-5x^2-2x\right)}{\left(2x^3+x^2\right)} = \tfrac{1}{2}\tfrac{\left(2x^3+x^2\right)-\left(6x^2+2x\right)}{\left(2x^3+x^2\right)} = \tfrac{1}{2}\tfrac{\left(2x^3+x^2\right)}{\left(2x^3+x^2\right)} - \tfrac{1}{2}\tfrac{\left(6x^2+2x\right)}{\left(2x^3+x^2\right)} = \tfrac{1}{2} - \tfrac{(3x+1)}{\left(2x^2+x\right)}$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{e^x}{2x^3+x^2}}\right)\right] = \frac{1}{2}\frac{\left(2x^3-5x^2-2x\right)}{\left(2x^3+x^2\right)}$

Or: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{e^x}{2x^3+x^2}}\right)\right] = \frac{1}{2} - \frac{(3x+1)}{\left(2x^2+x\right)}$

3. Compute: $\int e^{\left(6x^5+5x^2\right)}\left(3x^4+x\right)\,dx =$

1. Is u-sub appropriate?

   1. a. Is there a composite function?

      Yes. $e^{\left(6x^5+5x^2\right)}$

      Let $u = 6x^5 + 5x^2$

      b. Is there an "approximate function/derivative pair"?

      Yes. $\underbrace{\left(6x^5+5x^2\right)}_{\text{function}} \rightarrow \underbrace{\left(3x^4+x\right)}_{\text{deriv}}$

      Let $u = \left(6x^5+5x^2\right)$

      c. Do both criteria a and b suggest the same choice of u ?

      Yes!

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & 6x^5+5x^2 \\ \Rightarrow & \frac{du}{dx} & = & 30x^4+10x \\ \Rightarrow & du & = & \left(30x^4+10x\right)dx \\ \Rightarrow & \frac{1}{10}du & = & \left(3x^4+x\right)dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{e^{\left(6x^5+5x^2\right)}}_{e^u}\underbrace{\left(3x^4+x\right)\,dx}_{\frac{1}{10}du} = \int e^u \tfrac{1}{10}du = \tfrac{1}{10}\int e^u\,du$$

4. Integrate in terms of u

$$\tfrac{1}{10}\int e^u\,du = \tfrac{1}{10}e^u + C$$

5. Re-write in terms of x

$$\int e^{\left(6x^5+5x^2\right)}\left(3x^4+x\right)\,dx = \underbrace{\frac{1}{10}e^{\left(6x^5+5x^2\right)}+C}_{\frac{1}{10}e^u+C}$$

$$\boxed{\text{i.e., } \int e^{\left(6x^5+5x^2\right)}\left(3x^4+x\right)\,dx = \tfrac{1}{10}e^{\left(6x^5+5x^2\right)}+C}$$

4. Compute: $\int \frac{2x^2-1}{(4x^3-6x+3)^3}\,dx = \int \frac{1}{(4x^3-6x+3)^3}\left(2x^2-1\right)dx = \int \left(4x^3-6x+3\right)^{-3}\left(2x^2-1\right)dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes. $\left(4x^3-6x+3\right)^{-3}$

   Let $u = \left(4x^3-6x+3\right)$ i.e., "Let u = the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

   Yes. $\underbrace{\left(4x^3-6x+3\right)}_{\text{function}} \rightarrow \underbrace{\left(2x^2-1\right)}_{\text{deriv}}$

   Let $u = \left(4x^3-6x+3\right)$ i.e., "Let u = 'the function'"

   c. Do both criteria a and b suggest the same choice of u ?

   Yes!

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & 4x^3-6x+3 \\ \Rightarrow & \frac{du}{dx} & = & 12x^2-6 \\ \Rightarrow & du & = & \left(12x^2-6\right)dx \\ \Rightarrow & \frac{1}{6}du & = & \left(2x^2-1\right)dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\left(4x^3-6x+3\right)^{-3}}_{u^{-3}}\underbrace{\left(2x^2-1\right)dx}_{\frac{1}{6}du} = \int u^{-3}\tfrac{1}{6}du = \tfrac{1}{6}\int u^{-3}\,du$$

4. Integrate in terms of u

$$\tfrac{1}{6}\int u^{-3}\,du = \tfrac{1}{6}\tfrac{u^{-2}}{-2} + C = -\tfrac{1}{12}u^{-2} + C$$

5. Re-write in terms of x

$$\int \frac{2x^2-1}{(4x^3-6x+3)^3}\,dx = \underbrace{-\frac{1}{12}\left(4x^3-6x+3\right)^{-2} + C}_{-\frac{1}{12}u^{-2}+C}$$

$$\boxed{\text{i.e., } \int \frac{2x^2-1}{(4x^3-6x+3)^3}\,dx = -\tfrac{1}{12}\left(4x^3-6x+3\right)^{-2} + C}$$

5. Compute: $\int \frac{x^2+2x+1}{(x^3+3x^2+3x)}\,dx = \int \frac{1}{(x^3+3x^2+3x)}\left(x^2+2x+1\right)\,dx$

1. Is u-sub appropriate?

a. Is there a composite function?

Yes. $\frac{1}{(x^3+3x^2+3x)} = \left(x^3+3x^2+3x\right)^{-1}$

Let $u = \left(x^3+3x^2+3x\right)$ i.e., "Let $u =$ the 'inner' function"

b. Is there an "approximate function/derivative pair"?

Yes. $\underbrace{\left(x^3+3x^2+3x\right)}_{\text{function}} \to \underbrace{\left(x^2+2x+1\right)}_{\text{deriv}}$

Let $u = \left(x^3+3x^2+3x\right)$ i.e., "Let $u =$ 'the function'"

c. Do both criteria a and b suggest the same choice of u ?

Yes!

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & x^3+3x^2+3x \\ \Rightarrow & \frac{du}{dx} & = & 3x^2+6x+3 \\ \Rightarrow & du & = & \left(3x^2+6x+3\right)dx \\ \Rightarrow & \frac{1}{3}du & = & \left(x^2+2x+1\right)dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\frac{1}{(x^3+3x^2+3x)}}_{\frac{1}{u}} \underbrace{\left(x^2+2x+1\right)\,dx}_{\frac{1}{3}du} = \int \frac{1}{u}\frac{1}{3}du = \frac{1}{3}\int \frac{1}{u}\,du$$

4. Integrate in terms of u

$\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C$

5. Re-write in terms of x

$$\int \frac{x^2+2x+1}{(x^3+3x^2+3x)}\,dx = \underbrace{\frac{1}{3}\ln\left|x^3+3x^2+3x\right| + C}_{\frac{1}{3}\ln|u|+C}$$

$$\boxed{\text{i.e., } \int \frac{x^2+2x+1}{(x^3+3x^2+3x)}\,dx = \frac{1}{3}\ln\left|x^3+3x^2+3x\right| + C}$$

6. Compute: $\frac{d}{dx}\left[\arcsin\left(\cot\left(x\right)\right)\right] =$

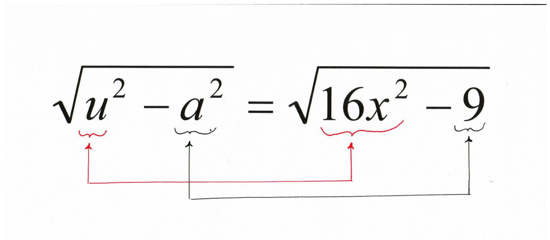
$$\underbrace{\frac{d}{dx}\left[\arcsin\left(\cot\left(x\right)\right)\right]}_{\frac{d}{dx}[\arcsin(u)]} = \underbrace{\frac{1}{\sqrt{1-\left(\cot\left(x\right)\right)^2}}}_{\frac{1}{\sqrt{1-u^2}}} \cdot \underbrace{\left(-\csc^2\left(x\right)\right)}_{\frac{du}{dx}} = \frac{-\csc^2(x)}{\sqrt{1-\cot^2(x)}} = -\frac{\csc^2(x)}{\sqrt{1-\cot^2(x)}}$$

i.e., $\frac{d}{dx}\left[\arcsin\left(\cot\left(x\right)\right)\right] = -\frac{\csc^2(x)}{\sqrt{1-\cot^2(x)}}$

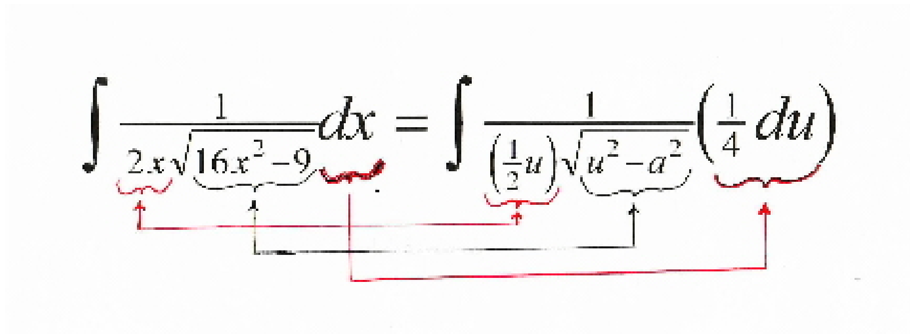
7. Compute: $\int \frac{1}{2x\sqrt{16x^2-9}}\,dx =$

   This appears to fit the form: $\int \frac{1}{u\sqrt{u^2-a^2}}du$

   If our conjecture is correct, then $\sqrt{u^2-a^2} = \sqrt{16x^2-9}$



$$\begin{array}{rl} & a^2 = 9 \\ \Rightarrow & a = 3 \\ & u^2 = 16x^2 \\ \Rightarrow & u = 4x \\ \Rightarrow & \frac{du}{dx} = 4 \\ \Rightarrow & du = 4dx \\ \Rightarrow & \frac{1}{4}du = dx \\ \text{Also:} & u = 4x \\ \Rightarrow & \frac{1}{2}u = 2x \\ \Rightarrow & \frac{1}{4}u = x \end{array}$$



3. Analyze in terms of u and du.

   $$\int \frac{1}{2x\sqrt{16x^2-9}}\,dx = \int \frac{1}{2x\sqrt{(4x)^2-3^2}}dx = \int \frac{1}{\left(\frac{1}{2}u\right)\sqrt{u^2-a^2}}\left(\frac{1}{4}du\right) = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-a^2}}du$$

4. Integrate

   $$\frac{1}{2}\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{2}\left[\frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)\right] + C = \frac{1}{2a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$$

5. Re-express in terms of x

   $$\int \frac{1}{2x\sqrt{16x^2-9}}\,dx = \underbrace{\frac{1}{2\,(3)}\operatorname{arcsec}\left(\frac{|4x|}{3}\right) + C}_{\frac{1}{2a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)+C} = \frac{1}{6}\operatorname{arcsec}\left(\frac{|4x|}{3}\right) + C$$

   $$\boxed{\int \frac{1}{2x\sqrt{16x^2-9}}\,dx = \frac{1}{6}\operatorname{arcsec}\left(\frac{|4x|}{3}\right) + C}$$

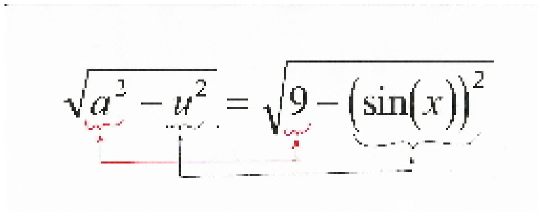
8. Compute: $\frac{d}{dx}\left[\tan^{-1}\left(e^x\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\tan^{-1}\left(e^x\right)\right]}_{\frac{d}{dx}[\tan^{-1}(u)]} = \underbrace{\frac{1}{1+\left(e^x\right)^2}}_{\frac{1}{1+u^2}} \cdot \underbrace{e^x}_{\frac{du}{dx}} = \frac{e^x}{(1+e^{2x})}$$

i.e., $\frac{d}{dx}\left[\tan^{-1}\left(e^x\right)\right] = \frac{e^x}{(1+e^{2x})}$

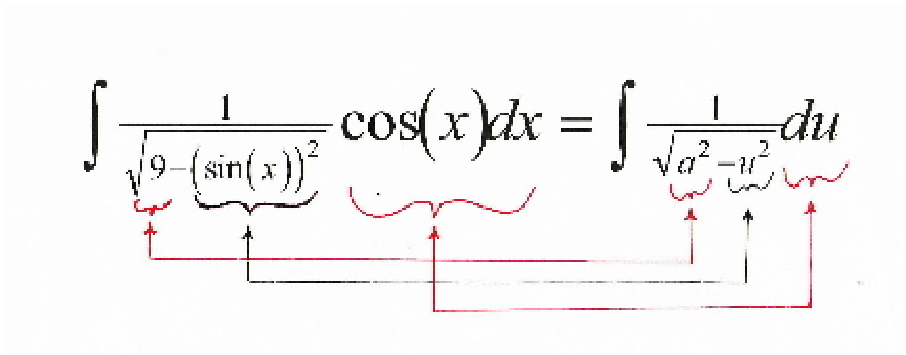
9. Compute: $\int \frac{1}{\sqrt{9-\sin^2(x)}} \cos(x)\, dx = \int \frac{1}{\sqrt{9-(\sin(x))^2}} \cos(x)\, dx$

$\int \frac{1}{\sqrt{9-(\sin(x))^2}} \cos(x)\, dx$ compare to: $\int \frac{1}{\sqrt{a^2-u^2}} du$



If this comparison is correct, then:

$$\begin{aligned} & a^2 = 9 \\ \Rightarrow\ & a = 3 \\ & u^2 = (\sin(x))^2 \\ \Rightarrow\ & u = \sin(x) \\ \Rightarrow\ & \tfrac{du}{dx} = \cos(x) \\ \Rightarrow\ & du = \cos(x)\, dx \end{aligned}$$



Now analyze the integral in terms of u and du.

$\int \frac{1}{\sqrt{9-\sin^2(x)}} \cos(x)\, dx = \int \frac{1}{\sqrt{9-(\sin(x))^2}} \cos(x)\, dx = \int \frac{1}{\sqrt{a^2-u^2}} du =$

3. Integrate:

$\int \frac{1}{\sqrt{a^2-u^2}} du = \arcsin\left(\frac{u}{a}\right) + C = \arcsin\left(\frac{\sin(x)}{3}\right) + C$

i.e., $\int \frac{1}{\sqrt{9-\sin^2(x)}} \cos(x)\, dx = \arcsin\left(\frac{\sin(x)}{3}\right) + C$

10. $z = \sec\left(\arctan\left(\frac{3x}{2}\right)\right)$ Re-write this equation as an equivalent algebraic equation.

Let $w = \arctan\left(\frac{3x}{2}\right)$

Then "w is the angle whose tangent is $\frac{3x}{2}$."

i.e., $\tan(w) = \frac{3x}{2}$

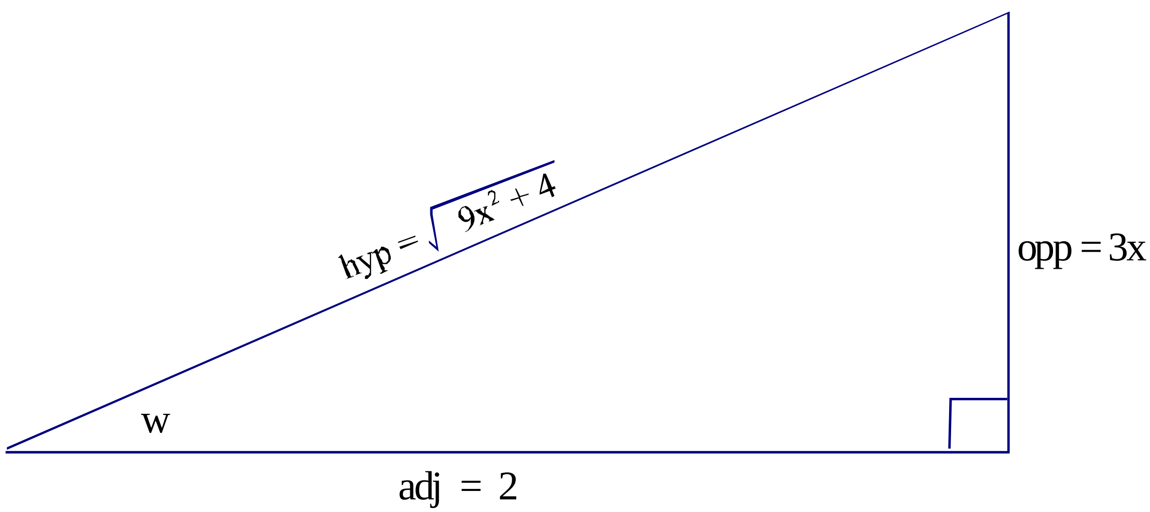
Draw a right triangle that depicts this relationship.

i.e., $\tan(w) = \frac{3x}{2} = \frac{\text{opp}}{\text{adj}} = \frac{3x}{2}$

$hyp^2 = opp^2 + adj^2 = (3x)^2 + 2^2 = 9x^2 + 4$

i.e., $hyp^2 = 9x^2 + 4$

$\Rightarrow hyp = \sqrt{9x^2 + 4}$



We want $z = \sec\left(\arctan\left(\frac{3x}{2}\right)\right)$

But since $w = \arctan\left(\frac{3x}{2}\right)$,

$\Rightarrow z = \sec(w)$

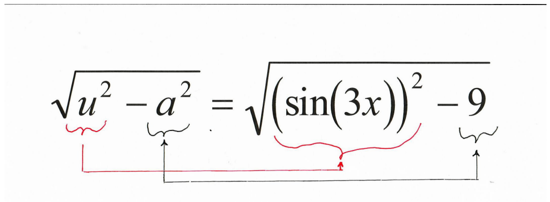
$\Rightarrow z = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{9x^2+4}}{2}$

i.e., $z = \frac{\sqrt{9x^2+4}}{2}$

Extra: Wow! 10 points (All or nothing)

Compute: $\int \frac{\cot(3x)}{\sqrt{\sin^2(3x)-9}}dx = \int \frac{1}{\sqrt{(\sin(3x))^2-9}} \cot(3x)\, dx$

$\frac{1}{\sqrt{(\sin(3x))^2-9}} \cot(3x)\, dx$ compare to: $\int \frac{1}{u\sqrt{u^2-a^2}}du$

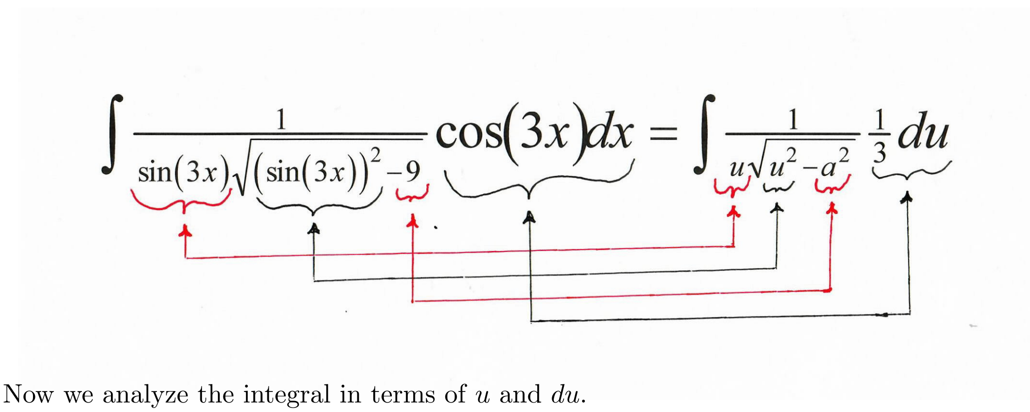


If this comparison is correct, then:

$$\begin{array}{ll} & a^2 = 9 \\ \Rightarrow & a = 3 \\ & u^2 = (\sin(3x))^2 \\ \Rightarrow & u = \sin(3x) \\ \Rightarrow & \frac{du}{dx} = 3\cos(x) \\ \Rightarrow & du = 3\cos(x)\,dx \\ \Rightarrow & \frac{1}{3}du = \cos(x)\,dx \end{array}$$

And our integral can be rearranged as follows:

$$\int \frac{\cot(3x)}{\sqrt{\sin^2(3x)-9}}dx = \int \frac{1}{\sqrt{(\sin(3x))^2-9}} \cot(3x)\, dx = \int \frac{1}{\sqrt{(\sin(3x))^2-9}} \frac{\cos(3x)}{\sin(3x)}dx = \int \frac{1}{\sin(3x)\sqrt{(\sin(3x))^2-9}} \cos(3x)\, dx$$



3. Now we analyze the integral in terms of u and du.

$$\int \frac{\cot(3x)}{\sqrt{\sin^2(3x)-9}}dx = \int \frac{1}{\sin(3x)\sqrt{(\sin(3x))^2-9}} \cos(3x)\, dx = \int \frac{1}{u\sqrt{u^2-a^2}} \frac{1}{3}du = \frac{1}{3}\int \frac{1}{u\sqrt{u^2-a^2}}du$$

4. Integrate in terms of u

$$\frac{1}{3}\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{3}\frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C$$

5. Re-write in terms of x

$$\int \frac{\cot(3x)}{\sqrt{\sin^2(3x)-9}}dx = \underbrace{\frac{1}{9}\sec^{-1}\left(\frac{|\sin(3x)|}{3}\right) + C}_{\frac{1}{3}\frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right)+C}$$

i.e., $\int \frac{\cot(3x)}{\sqrt{\sin^2(3x)-9}}dx = \frac{1}{9}\sec^{-1}\left(\frac{|\sin(3x)|}{3}\right) + C$