MTH 4436 Homework Set 4.3; p. 73 2-7

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2. Prove the following statements:

   (a) For any integer a, the units digit of a^2 is $0, 1, 4, 5, 6,$ or 9.

   The key observation here, is that the units digit of a^2 is congruent to $a^2 \pmod{10}$.

   Also, if m is the units digit of n, then $\exists$ a natural number k, such that $n = 10k+m$. For example, $n = 1295$ can be written as

$$n = (10)(129) + 5$$

   At any rate, the units digit of a^2 is equal to $a^2 \pmod{10} \equiv (10k+m)^2 \pmod{10} \equiv m^2 \pmod{10}$.

   There are 10 cases:

$m =$	units digit $= m^2 \pmod{10}$
0	0
1	1
2	4
3	9
4	6
5	5
6	6
7	9
8	4
9	1

   Thus, the units digit of a^2 is either $0, 1, 4, 5, 6,$ or 9.

(b) Any one of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ can occur as the units digit of a^3.

The key observation here, is that the units digit of a^3 is congruent to $a^3 \pmod{10}$.

Also, if m is the units digit of n, then $\exists$ a natural number k, such that $n = 10k+m$.

The units digit of a^3 is equal to $a^3 \pmod{10} \equiv (10k+m)^3 \pmod{10} \equiv m^3 \pmod{10}$.

There are 10 cases:

$m =$	units digit $= m^3 \pmod{10}$
0	0
1	1
2	8
3	7
4	4
5	5
6	6
7	3
8	2
9	9

Thus, the units digit of a^3 can be any one of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

(c) For any integer a, the units digit of a^4 is $0, 1, 5$, or 6.

Again, the key observation here, is that the units digit of a^4 is congruent to $a^4 \pmod{10}$.

Also, if m is the units digit of n, then $\exists$ a natural number k, such that $n = 10k+m$.

The units digit of a^4 is equal to $a^4 \pmod{10} \equiv (10k+m)^4 \pmod{10} \equiv m^4 \pmod{10}$.

There are 10 cases:

$m =$	units digit $= m^4 \pmod{10}$
0	0
1	1
2	6
3	1
4	6
5	5
6	6
7	1
8	6
9	1

Thus, the units digit of a^4 is either $0, 1, 5$, or 6.

(d) The units digit of a triangular number is either $0, 1, 3, 5, 6$, or 8.

Let t_n be the n^{th} triangular number. As before, the units digit of t_n is congruent to $t_n \pmod{10}$.

By problem 1a, in set 1.1, $t_n = \frac{n(n+1)}{2}$. i.e. $t_n = \frac{n^2+n}{2}$

m	$m^2 \pmod{10}$	$(m^2+m) \pmod{10}$	$\frac{(m^2+m)}{2} \pmod{10}$
0	0	0	0, 5
1	1	2	1, 6
2	4	6	3, 8
3	9	2	1, 6
4	6	0	0, 5
5	5	0	0, 5
6	6	2	1, 6
7	9	6	3, 8
8	4	2	1, 6
9	1	0	0, 5

Thus, the units digit of a triangular number is either $0, 1, 3, 5, 6$, or 8.

3. Find the last two digits of the number 9^{9^9}. [Hint: $9^9 \equiv 9 \pmod{10}$; hence, $9^{9^9} = 9^{9+10k}$; now use the fact that $9^9 \equiv 89 \pmod{100}$]

First, let's verify these hints:

$9^9 \equiv 9 \pmod{10}$

Observe: $9 \equiv -1 \pmod{10}$

Hence, $9^9 \equiv (-1)^9 \pmod{10} \equiv -1 \pmod{10} \equiv 9 \pmod{10}$

i.e., $9^9 \equiv 9 \pmod{10}$

$9^{9^9} = 9^{9+10k}$

Observe: $9^9 \equiv 9 \pmod{10} \Rightarrow 9^9 - 9 = 10k$, for some $k \in \mathbb{Z}, \Rightarrow 9^9 = 9 + 10k$, for some $k \in \mathbb{Z}$

i.e., $9^9 = 9 + 10k$

Hence, $9^{9^9} = 9^{9+10k}$

$$\boxed{9^9 \equiv 89 \pmod{100}}$$

I don't think that there's a really clever way to do this:

Observe: $9^2 = 81 \cdot 9 = 729 \cdot 9 \equiv 29 \cdot 9 = 261 \equiv 61 \cdot 9 = 549$

$9^3 = 9^2 \cdot 9 = 81 \cdot 9 \equiv (-19) \cdot 9 \pmod{100} \equiv (-171) \pmod{100} \equiv 29 \pmod{100}$

i.e. $9^3 \equiv 29 \pmod{100}$

$9^6 = \left(9^3\right)^2 \equiv (29)^2 \pmod{100} \equiv 841 \pmod{100} \equiv 41 \pmod{100}$

i.e., $9^6 \equiv 41 \pmod{100}$

$9^9 = 9^3 9^6 \equiv (29)\,(41) \pmod{100} \equiv 1189 \pmod{100} \equiv 89 \pmod{100}$

i.e., $9^9 \equiv 89 \pmod{100}$

Now . . . Back to our exercise!!!

Find the last two digits of the number 9^{9^9}

Observe:

$9^{9^9} = 9^{9+10k} = 9^9 \cdot 9^{10k} = 9^9 \cdot \left(9^{10}\right)^k = 9^9 \cdot \left(9^9 \cdot 9\right)^k \equiv 9^9 \cdot \left((89)\,(9)\right)^k \pmod{100}$

$\equiv 9^9 \cdot \left((-11)\,(9)\right)^k \pmod{100} \equiv 9^9 \cdot \left((1)\right)^k \pmod{100} \equiv 9^9 \cdot 1 \pmod{100} \equiv 9^9 \pmod{100}$

$\equiv 89 \pmod{100}$

i.e., $9^{9^9} \equiv 89 \pmod{100}$

Hence, the last two digits of 9^{9^9} are 89.

4. Without performing the divisions, determine whether the integers $176,521,221$ and $149,235,678$ are divisible by 9 or 11.

$$\boxed{176,521,221}$$

Observe: $1+7+6+5+2+1+2+2+1=27$, which is divisible by 9.

Since $9|\left(1+7+6+5+2+1+2+2+1\right)$, $9|176,521,221$ also.

Note also, that $1-2+2-1+2-5+6-7+1=-3$, which is NOT divisible by 11.

Since $11\nmid\left(1-2+2-1+2-5+6-7+\right)$, it follows that $11\nmid 176,521,221$.

$$\boxed{149,235,678}$$

Observe: $1+4+9+2+3+5+6+7+8=45$, which is divisible by 9.

Since $9|\left(1+4+9+2+3+5+6+7+8\right)$, $9|149,235,678$ also.

Note also, that $8-7+6-5+3-2+9-4+1=9$ which is NOT divisible by 11.

Since $11\nmid\left(8-7+6-5+3-2+9-4+1\right)$, it follows that $11\nmid 176,521,221$.

5. ~

(a) Obtain the following generalization of Theorem 4.5: If the integer $\mathbf{N}$ is represented in base b by

$$\mathbf{N} = a_m b^m + a_{m-1} b^{m-1} + \ldots + a_2 b^2 + a_1 b^1 + a_0 \qquad 0 \leq a_k \leq b - 1$$

Then $(b-1)\,|\mathbf{N}$ if and only if $(b-1)\,|S$, where $S = (a_m + a_{m-1} + \ldots + a_2 + a_1 + a_0)$.

Proof. Consider the polynomial with integer coefficients, $P(x) = \sum_{k=0}^{m} a_k x^k$.

Observe: $b \equiv 1 \,(\mathrm{mod}\,(b-1))$.

Hence, $P(b) \equiv P(1) \,(\mathrm{mod}\,(b-1))$.

But here's the catch:

$$\begin{aligned} P(b) &= a_m b^m + a_{k-1} b^{m-1} + \ldots + a_2 b^2 + a_1 b^1 + a_0 b^0 \\ &= N \end{aligned}$$

and

$$\begin{aligned} P(1) &= a_m (1)^m + a_{m-1} (1)^{m-1} + \ldots + a_2 (1)^2 + a_1 (1)^1 + a_0 (1)^0 \\ &= a_m + a_{m-1} + \ldots + a_2 + a_1 + a_0 \\ &= S \end{aligned}$$

Now suppose that $(b-1)\,|N$.

Since, $N = P(b)$, this is true if and only if $(b-1)\,|P(b)$.

$\Leftrightarrow P(b) \equiv 0 \,(\mathrm{mod}\,(b-1))$.

Since, $P(b) \equiv P(1) \,(\mathrm{mod}\,(b-1))$, this is true if and only if $P(1) \equiv 0 \,(\mathrm{mod}\,(b-1))$,

Which, in turn, is true if and only if $(b-1)\,|P(1)$,

and this is true if and only if $9|S$. ■

(b) Give criteria for the divisibility of N by 3 and 8 that depend on the digits of N when written in base 9.

$\boxed{3}$

Given that $\mathbf{N} = a_m 9^m + a_{m-1} 9^{m-1} + \ldots + a_2 9^2 + a_1 9^1 + a_0$,

observe that $(a_m 9^m + a_{m-1} 9^{m-1} + \ldots + a_2 9^2 + a_1 9^1) \equiv 0 \pmod 3$.

Hence, $\mathbf{N} = (a_m 9^m + a_{m-1} 9^{m-1} + \ldots + a_2 9^2 + a_1 9^1 + a_0) \equiv a_0 \pmod 3$.

i.e., $\mathbf{N} \equiv a_0 \pmod 3$.

Observe:

$3|N \Leftrightarrow \mathbf{N} \equiv 0 \pmod 3 \Leftrightarrow a_0 \equiv 0 \pmod 3 \Leftrightarrow 3|a_0$.

Hence, $3|N$ if and only if $3|a_0$.

$\boxed{8}$

If we let $b = 9$, then $(b-1) = 8$.

By part a, If the integer $\mathbf{N}$ is represented in base 9 by

$$\mathbf{N} = a_m 9^m + a_{m-1} 9^{m-1} + \ldots + a_2 9^2 + a_1 9^1 + a_0 \qquad 0 \le a_k \le 8$$

Then $8|\mathbf{N}$ if and only if $8|(a_m + a_{m-1} + \ldots + a_2 + a_1 + a_0)$.

(c) Is the number $(447836)_9$ divisible by 3 and 8?

$\boxed{3}$

$a_0 = 6$.

Since $3|a_0$, it follows that $3|(447836)_9$

$\boxed{8}$

Observe: $4+4+7+8+3+6 = 32_{10}$

Since $8|32_{10}$ it follows that $8|(447836)_9$

6. Working modulo 9 or 11, find the missing digits in the calculations below:

(a) $51840 \cdot 273581 = 1418243x040$.

Observe: $9|\left(5+1+8+4+0\right)$, hence, $9|\left(51840\right)$.

Since $51840 \cdot 273581 = 1418243x040$, it follows that $9|\left(1418243x040\right)$.

Hence, $9|\left(1+4+1+8+2+4+3+x+0+4+0\right) \Rightarrow 9|\left(27+x\right) \Rightarrow 9|x$

$\Rightarrow$ either $x = 0$ or $x = 9$

Unfortunately, in this case, the "divisibility by 9 criterion" turns out to be inconclusive.

OK, so let's try using the "divisibility by 11 criterion."

Observe: $11|\left(1-8+5-3+7-2\right)$, hence $11|273581$.

Since $51840 \cdot 273581 = 1418243x040$ and 11 divides a factor of the Left Hand side of the equation, it follows that 11 divides the Right Hand side of the equation.

i.e., $11|\left(1418243x040\right)$.

Hence, $11|\left(0-4+0-x+3-4+2-8+1-4+1\right) \Rightarrow 11|\left(-13-x\right) \Rightarrow 11|\left(13+x\right)$

$\Rightarrow x = 9$

(b) $2x99561 = \left[3\left(523+x\right)\right]^2$.

Observe: $2x99561 = \left[3\left(523+x\right)\right]^2 = 9\left(523+x\right)^2$

The point of this observation is that $9|2x99561$.

Hence, $9|\left(2+x+9+9+5+6+1\right) \Rightarrow 9|\left(32+x\right) \Rightarrow x = 4$.

Let's check: $\left[3\left(523+4\right)\right]^2 = 1581^2 = 2499561$.

Check!

(c) $2784x = x \cdot 5569$

At first glance, it appears that we won't be able to "find the missing digit, working modulo 9 or 11."

If we try applying the "divisibility by 9 criterion," we see that, in order for $2784x$ to be divisible by 9, x must be equal to 6. However, since $(5+5+6+9) = 25$, no value of x, other than $x = 9$, will make $x \cdot 5569$ divisible by 9.

If we try applying the "divisibility by 11 criterion" to the right hand side of the equation:

$$x \cdot 5569$$

we see that, $(9-6+5-5) = 3$.

Hence, 5569 does not have a factor of 11. Therefore, the only value of x that will make $x \cdot 5569$ divisible by 11 is $x = 11$.

Since, from the context of this exercise (see the left hand side of the equation), x is a digit, the conclusion that $x = 11$ is an impossibility.

Thus, the "divisibility by 11 criterion" does not seem to be applicable either.

(d) in order for $2784x$ to be divisible by 11, $11\,(x-4+8-7+2) \Rightarrow x = 1$.

However, since $(9-6+5-5) = 3$, it follows that 5569 does not have a factor of 11. Therefore, the only value of x that will make $x \cdot 5569$ divisible by 11 is $x = 11$. no value of x, other than $x = 9$, will make $x \cdot 5569$ divisible by 9.

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Let's concentrate on the units digit.

$2784x = x \cdot 5569 \Rightarrow x \cdot 9 = *x$

A consideration of the non-zero possibilities shows that $x = 5$:

$$\begin{aligned}
1 \cdot 9 &= 9 \\
2 \cdot 9 &= 18 \\
3 \cdot 9 &= 27 \\
4 \cdot 9 &= 36 \\
5 \cdot 9 &= 45 \\
6 \cdot 9 &= 54 \\
7 \cdot 9 &= 63 \\
8 \cdot 9 &= 72 \\
9 \cdot 9 &= 81
\end{aligned}$$

Check: $5 \cdot 5569 = 27845$

Check!

(e) $512 \cdot 1x53125 = 1,000,000,000$

(For future reference, note that $512 = 2^9$. So the only prime factor of 512 is 2.)

Initially, it appears that neither the "divisibility by 9 criterion" nor the "divisibility by 11 criterion" apply here, as

$9 \nmid (1+0+0+0+0+0+0+0+0+0)$ and

$11 \nmid (0-0+0-0+0-0+0-0+0-1)$.

Perhaps we can manipulate the equation algebraically, so that either the "divisibility by 9 criterion" or the "divisibility by 11 criterion" DOES apply.

Since $9 \nmid (1,000,000,000)$ and $11 \nmid (1,000,000,000)$, we might try to remedy this situation algebraically by adding/subtracting a constant to/from each side of the equation so that the right hand side will be divisible by either 9 or 11.

Our first attempt might be to subtract 1 from the right-hand side of the equation, which will make the right-hand side equal to 999,999,999 – a number that is clearly divisible by 9.

But when we attempt to subtract 1 from the left-hand side of the equation, we have a problem:

$512 \cdot 1x53125 - 1$ cannot be expressed as a product, so we can't apply the divisibility criteria for 9 and 11 to the left-hand side of the equation.

The lesson that we learn from this is that we must augment/decrement the right-hand side of the equation in increments of 512.

For example:

$512 \cdot 1x53125 + 512 = 1,000,000,000 + 512$

$\Leftrightarrow 512 \cdot (1x53125 + 1) = 1,000,000,512$

$\Leftrightarrow 512 \cdot 1x53126 = 1,000,000,512$

i.e., each increment/decrement of the factor $1x53125$ by 1 on the left-hand side translates into a corresponding increment/decrement of the right-hand side by $1,000,000,000$ by 512.

Sooooo . . . incrementing both sides in this fashion yields:

$512 \cdot 1x53126 = 1,000,000,512$

The right-hand side is clearly divisible by 9, as $9 | (1+0+0+0+0+0+0+5+1+2)$

Hence, $9 | (512 \cdot 1x53126) \Rightarrow 9 | 1x53126$, since $9 \nmid 512$.

$\Rightarrow 9 | (1+x+5+3+1+2+6) \Rightarrow 9 | (x+18) \Rightarrow x = 0$ or $x = 9$.

The divisibility by 9 criterion proves to be inconclusive. 

Sooooo . . . let's increment both sides again. This will yield:

$512 \cdot 1x53127 = 1,000,001,024$

The right-hand side is clearly divisible by 11, as $11 | (4 - 2 + 0 - 1 + 0 - 0 + 0 - 0 + 0 - 1)$

Hence, $11 | (512 \cdot 1x53127) \Rightarrow 11 | 1x53127$, since $11 \nmid 512$.

$\Rightarrow 11 | (7 - 2 + 1 - 3 + 5 - x + 1) \Rightarrow 11 | (9 - x) \Rightarrow x = 9$

Check: $512 \cdot 1953125 = 2^9 \cdot 5^9 = (2 \cdot 5)^9 = 1,000,000,000$

Check!

$x = 9$.

7. Establish the following divisibility criteria:

(a) An integer is divisible by 2 if and only if its units digit is $0, 2$, 4, 6, or 8.

Proof. Let N be the integer under consideration, and let x be the units digit of N.

Note that N can be expressed as

$$N = k \cdot 10 + x$$

where $k \in \mathbf{Z}$, and $x = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Note that $k \cdot 10 \equiv 0 \pmod{2}$.

Hence, $N \equiv x \pmod{2}$.

Observe:

$2|N$

$\Leftrightarrow N \equiv 0 \pmod{2}$

$\Leftrightarrow x \equiv 0 \pmod{2}$

$\Leftrightarrow x \in \{0, 2, 4, 6, 8\}$. ■

(b) An integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

(i.e., if $N = a_m a_{m-1} \ldots a_2 a_1 a_0$, then $3|N \Leftrightarrow 3|(a_m + a_{m-1} + \ldots + a_2 + a_1 + a_0)$.)

Proof. Define $P(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_2 x^2 + a_1 x + a_0$.

Note: Since $10 \equiv 1 \pmod{3}$, it follows that $P(10) \equiv P(1) \pmod{3}$.

Observe:
$3|N \Leftrightarrow N \equiv 0 \pmod{3} \Leftrightarrow P(10) \equiv 0 \pmod{3} \Leftrightarrow P(1) \equiv 0 \pmod{3} \Leftrightarrow 3|P(1) \Leftrightarrow 3|(a_m + a_{m-1} + \ldots + a_2 + a_1 + a_0)$. ■

(c) An integer is divisible by 4 if and only if the number formed by its tens and units digits is divisible by 4.

Proof. Let $N = a_m a_{m-1} \dots a_2 a_1 a_0$. Note that N can be written as:

$$N = k \cdot 10^2 + a_1 a_0.$$

Since $k \cdot 10^2 \equiv 0 \pmod{4}$, it follows that $N = k \cdot 10^2 + a_1 a_0 \equiv a_1 a_0 \pmod{4}$.

i.e., $N \equiv a_1 a_0 \pmod{4}$.

Observe:

$4|N$

$\Leftrightarrow N \equiv 0 \pmod{4}$

$\Leftrightarrow a_1 a_0 \equiv 0 \pmod{4}$

$\Leftrightarrow 4|a_1 a_0$. ■

(d) An integer is divisible by 5 if and only if its units digit is 0 or 5.

Proof. Let N be the integer under consideration, and let x be the units digit of N.

Note that N can be expressed as

$$N = k \cdot 10 + x$$

where $k \in \mathbf{Z}$, and $x = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Note that $k \cdot 10 \equiv 0 \pmod{5}$.

Hence, $N \equiv x \pmod{5}$.

Observe:

$5|N$

$\Leftrightarrow N \equiv 0 \pmod{5}$

$\Leftrightarrow x \equiv 0 \pmod{5}$

$\Leftrightarrow x \in \{0, 5\}$. ■