**MTH 1125 Test #3 - Solutions**

Spring 2016

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**Show CLEARLY how you arrive at your answers.**

1. $f(x) = x^3 - 3x^2 - 9x + 2.$ Identify the intervals on which $f(x)$ is increasing/decreasing, and identify all relative maximums and minimums.

   i. Compute $f'(x)$ and find critical numbers

   $f'(x) = 3x^2 - 6x - 9$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 3x^2 - 6x - 9 = 0$

   $\Rightarrow 3x^2 - 6x - 9 = 0$

   $\Rightarrow x^2 - 2x - 3 = 0$

   $\Rightarrow (x + 1)(x - 3) = 0$

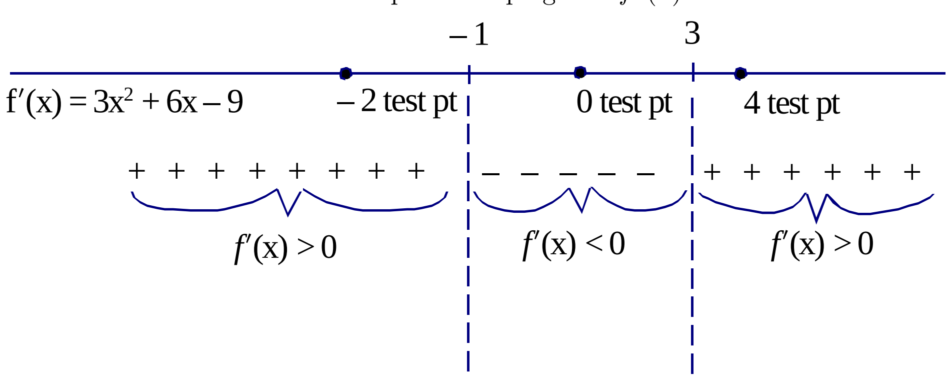
   $\Rightarrow x = -1;\ x = 3$ critical numbers

   b. "Type b" $(f'(c)$ undefined$)$

   There are none.

   ii. Draw a sign graph of $f'(x)$, using the critical numbers to partition the $x$-axis

   iii. From each interval select a "test point" to plug into $f'(x)$



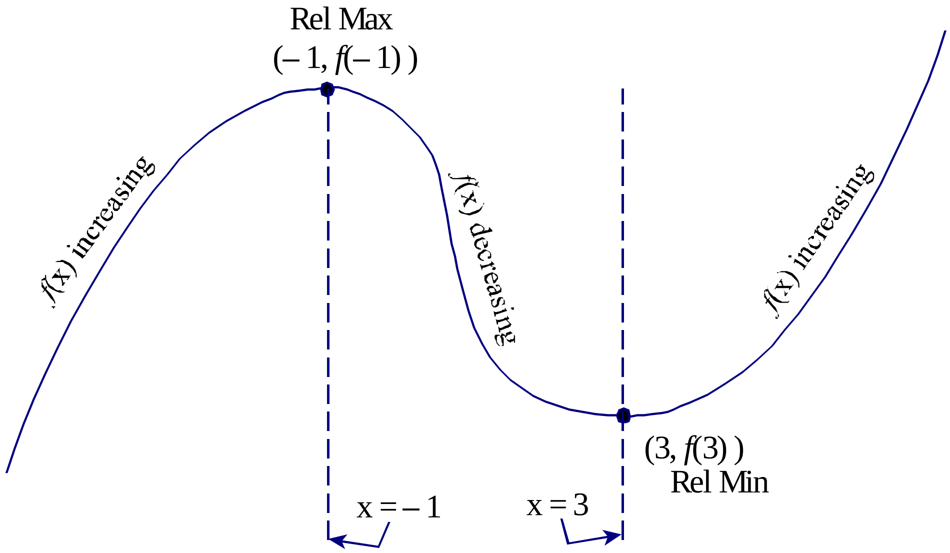

$f(x)$ is **increasing** on the intervals $(-\infty, -1)$ and $(3, \infty)$

(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval $(-1, 3)$

(Because $f'(x)$ is negative on this interval)

iv. To find the relative maxes and mins, sketch a rough graph of $f(x)$.




$f(x)$ is **increasing** on the intervals $(-\infty, -3)$ and $(1, \infty)$

$f(x)$ is **decreasing** on the interval $(-3, 1)$

$(-1, f(-1)) = (-1, 7)$ Relative Max

$(3, f(1)) = (3, -25)$ Relative Min

2. $f(x) = 2x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$. Identify the intervals on which $f(x)$ is increasing/decreasing, and identify all relative maximums and minimums.

i. Compute $f'(x)$ and find critical numbers

$f'(x) = \frac{10}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}} = \frac{10x^{\frac{2}{3}}}{3} - \frac{10}{3x^{\frac{1}{3}}} = \frac{10x^{\frac{2}{3}}}{3}\frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}} - \frac{10}{3x^{\frac{1}{3}}} = \frac{10x^{\frac{3}{3}}}{3} - \frac{10}{3x^{\frac{1}{3}}} = \frac{10x-10}{3x^{\frac{1}{3}}}$

i.e. $f'(x) = \frac{10x-10}{3x^{\frac{1}{3}}}$

a. "Type a" $(f'(c) = 0)$

Set $f'(x) = \frac{10x-10}{3x^{\frac{1}{3}}} = 0$

$\Rightarrow 10x - 10 = 0$

$\Rightarrow 10x = 10$

$\Rightarrow x = 1$ critical number

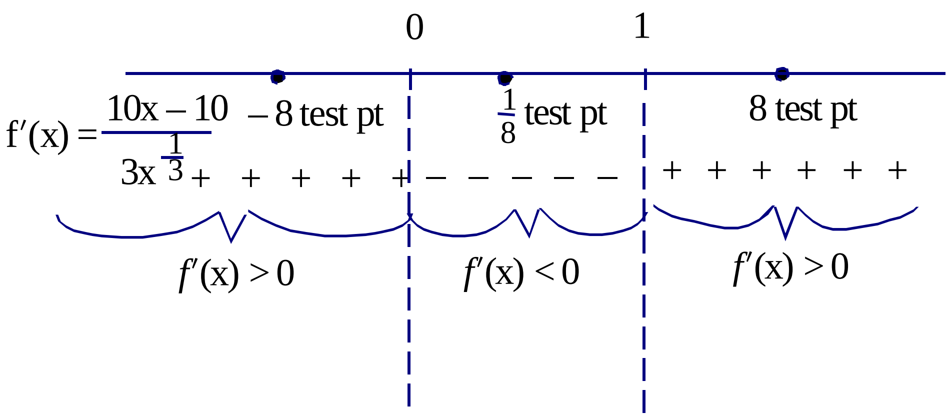
b. "Type b" ($f'(c)$ undefined)

Set denominator $3x^{\frac{1}{3}} = 0$

$\Rightarrow x = 0$ critical number

ii. Draw a sign graph of $f'(x)$, using the critical numbers to partition the $x$-axis

iii. From each interval select a "test point" to plug into $f'(x)$



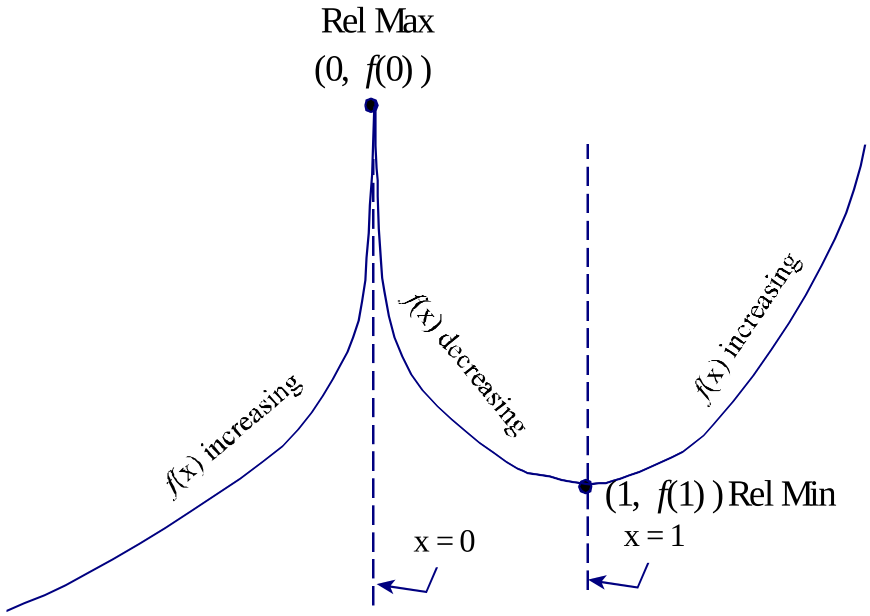

$f(x)$ is **increasing** on the intervals $(-\infty, 0)$ and $(1, \infty)$

(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the intervals and $(0, 1)$

(Because $f'(x)$ is negative on this interval)

iv. To find the relative maxes and mins, sketch a rough graph of $f(x)$.




$f(x)$ is **increasing** on the intervals and $(-\infty, 0)$ and $(1, \infty)$

$f(x)$ is **decreasing** on the interval $(0, 1)$

**Relative Max** $(0, f(0)) = (0, 0)$

**Relative Min** $(1, f(1)) = (1, -3)$

3. $y^3 + 3x^2y^4 = 5x^2 + \sin(y)$; Compute $y'$

i. Differentiate both sides w.r.t. $x$

$$\frac{d}{dx}\left[y^3 + 3x^2y^4\right] = \frac{d}{dx}\left[5x^2 + \sin(y)\right]$$

$$\Rightarrow 3y^2 \cdot y' + \frac{d}{dx}\left[\underbrace{3x^2}_{1^{st}}\underbrace{y^4}_{2^{nd}}\right] = 10x + \cos(y) \cdot y'$$

$$\Rightarrow 3y^2 \cdot y' + \left(\underbrace{6x}_{1^{st}\text{ prime}} \cdot \underbrace{y^4}_{2^{nd}} + \underbrace{4y^3 \cdot y'}_{2^{nd}} \cdot \underbrace{3x^2}_{1^{st}}\right) = 10x + \cos(y) \cdot y'$$

i.e., $3y^2y' + 6xy^4 + 4y^3y'3x^2 = 10x + \cos(y)\,y'$

ii. Solve algebraically for $y'$

a. Get $y'$ terms on left side, all other terms on right side

$\Rightarrow 3y^2y' + 4y^3y'3x^2 - \cos(y)\,y' = 10x - 6xy^4$

b. Factor out $y'$

$\Rightarrow \left(3y^2 + 4y^3 3x^2 - \cos(y)\right)\, y' = 10x - 6xy^4$

c. Divide both sides by the cofactor of $y'$

$y' = \frac{10x - 6xy^4}{3y^2 + 4y^3 3x^2 - \cos(y)}$

$$\boxed{y' = \frac{10x - 6xy^4}{3y^2 + 4y^3 3x^2 - \cos(y)}}$$

4. $f(x) = x^3 - 3x + 3$ on the interval $[-2, 2]$. Find the absolute maximum value and absolute minimum value of $f(x)$.

   Since $f(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[-2, 2]$, we can use the Absolute Max/Min Value Test

   i. Compute $f'(x)$ and find the critical numbers

      $f'(x) = 3x^2 - 3$

      a. "Type a" ($f'(c) = 0$)

         Set $f'(x) = 3x^2 - 3 = 0$

         $\Rightarrow 3x^2 - 3 = 0$

         $\Rightarrow x^2 - 1 = 0$

         $\Rightarrow (x + 1)(x - 1) = 0$

         $\Rightarrow x = -1$; $x = 1$ "type a" crit. numbers

      b. "Type b" ($f'(c)$ is undefined)

         No "type b" critical numbers.

   ii. Plug critical numbers and endpoints into the original function.

      $f(-2) = (-2)^3 - 3(-2) + 3 = 1 \leftarrow$ Abs Min Value

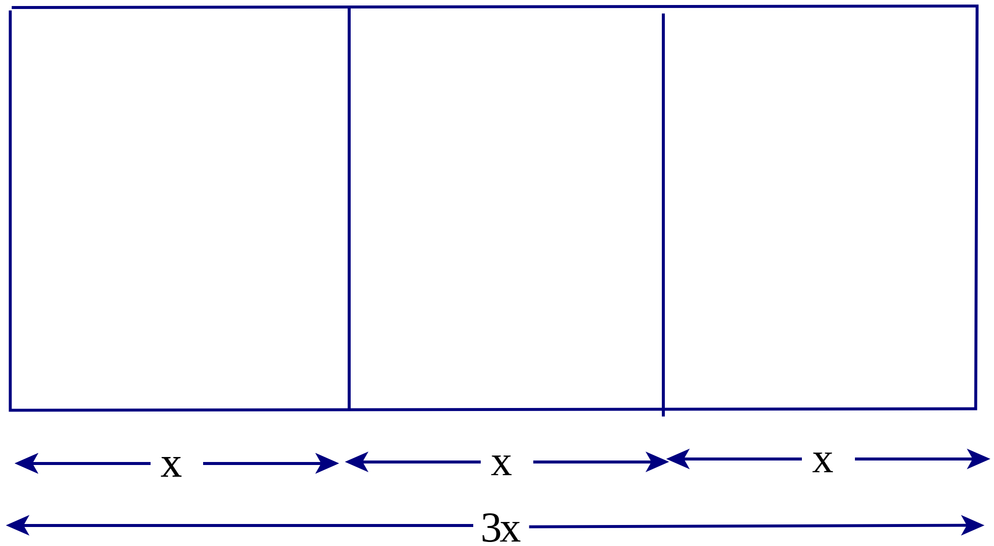
      $f(-1) = (-1)^3 - 3(-1) + 3 = 5 \leftarrow$ Abs Max Value

      $f(1) = (1)^3 - 3(1) + 3 = 1 \quad \leftarrow$ Abs Min Value

      $f(2) = (2)^3 - 3(2) + 3 = 5 \quad \leftarrow$ Abs Max Value

| |
|---|
| Abs Max Value $= 5$<br>(attained at $x = -1$ and $x = 2$)<br><br>Abs Min Value $= 1$<br>(attained at $x = -2$ and x = 1) |

5. Farmer Joe has 600 yards of fencing. He will use the fencing to construct a rectangular pen. He will use some of the fencing to partition the pen into three smaller pens, of similar shape and equal area (as shown below). What should the overall dimensions of the pen be in order for the pen to contain the largest area possible?




1. Determine the quantity to be maximized - Give it a name!

   Maximize the (Total) **Area** of the rectangular pen, $A = 3xy$

   **a.** Draw a picture where relevant.

   1. 1. (Done)

2. Express $A$ as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that Farmer Joe must use exactly 600 yds of fence.

   Hence, $6x + 4y = 600$ yds

   $\Rightarrow 4y = 600 \text{ yds} - 6x$

   $\Rightarrow y = 150 \text{ yds} - \frac{3}{2}x$

   Plug this into the equation $A = 3xy$

   $\Rightarrow A(x) = 3x\left(150 \text{ yds} - \frac{3}{2}x\right) = 450 \text{ yds } x - \frac{9}{2}x^2$

   i.e., $A(x) = 450 \text{ yds } x - \frac{9}{2}x^2$

3. Determine the restrictions on the independent variable $x$.

   From the picture, $0 \text{ yds} \leq x \leq \frac{600 \text{ yds}}{6}$

   $\Rightarrow 0 \text{ yds} \leq x \leq 100 \text{ yds}$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0 \text{ yds}, 100 \text{ yds}]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = 450 \text{ yds} - 9x$

      a. "Type a" $(f'(c) = 0)$

         $\Rightarrow A'(x) = 450 \text{ yds} - 9x = 0$

         $\Rightarrow 9x = 450 \text{ yds}$

         $\Rightarrow x = 50 \text{ yds}$ (crit. number)

      b. "Type b" ($f'(c)$ is undefined)

         Look for $x$-values that cause division by zero in $f'(x)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0 \text{ yds}) = 450 \text{ yds}\,(0 \text{ yds}) - \frac{9}{2}(0 \text{ yds})^2 = 0 \text{ yds}^2$

      $A(50 \text{ yds}) = 450 \text{ yds}\,(50 \text{ yds}) - \frac{9}{2}(50 \text{ yds})^2 = 11,250 \text{ yds}^2 \longleftarrow$ Abs Max Value

      $A(100 \text{ yds}) = 450 \text{ yds}\,(100 \text{ yds}) - \frac{9}{2}(100 \text{ yds})^2 = 0 \text{ yds}^2$

5. Make sure that we've answered the original question.

   "What should the overall dimensions of the pen be in order for the pen to contain the largest area possible?"

   $x = 50 \text{ yds}$

   Length $= 3x = 150 \text{ yds}$

   Width $= y = 150 \text{ yds} - \frac{3}{2}x = 150 \text{ yds} - \frac{3}{2}(50 \text{ yds}) = 75 \text{ yds}$

   | |
   |---|
   | Length $= 3x = 150 \text{ yds}$ |
   | Width $= y = 75 \text{ yds}$ |