MTH 4425 – Problem List For Exam #3 – Part #2

Spring 2021

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Definition 1 *A sequence* $\{a_n\}_{n=1}^{\infty}$ *is a* Cauchy Sequence *if* $\forall \varepsilon > 0$, $\exists\ N = N(\varepsilon) \in \mathbf{N}$ *such that* $m, n > N \Rightarrow |a_m - a_n| < \varepsilon$.

1. Every convergent sequence is a Cauchy sequence.

   Proof. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence, and let L be the limit of the sequence.

   Let $\varepsilon > 0$ be given.

   Since $\{a_n\}_{n=1}^{\infty}$ converges to L, $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $n > N \Rightarrow |a_n - L| < \frac{\varepsilon}{2}$.

   Thus, for $m, n > N$, we have $|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| =$

   $|a_m - L| + |a_n - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

   i.e., $m, n > N \Rightarrow |a_m - a_n| < \varepsilon$.

   Thus, $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. ■

   Scratchwork: For $m, n > N$, we want $|a_m - a_n| < \varepsilon$.

   In order to use the fact that $\{a_n\}_{n=1}^{\infty}$ converges to L, we somehow must work L into the inequality $|a_m - a_n| < \varepsilon$.

   So, we'll use the old "add and subtract" trick.

   $\Rightarrow |a_m - a_n| < \varepsilon \Rightarrow |a_m - L + L - a_n| < \varepsilon \Rightarrow |a_m - L + L - a_n| \leq$

   $|a_m - L| + |L - a_n| < \varepsilon \Rightarrow |a_m - L| + |a_n - L| < \varepsilon$

   Observe: if we make $|a_m - L| < \frac{\varepsilon}{2}$ and $|a_n - L| < \frac{\varepsilon}{2}$, then we'll have $|a_m - L + L - a_n| \leq$

   $|a_m - L| + |L - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

   Thus, we choose $N = N(\varepsilon) \in \mathbf{N}$ such that $n > N \Rightarrow |a_n - L| < \frac{\varepsilon}{2}$.

2. Every Cauchy Sequence has a limit point.

Proof. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Then $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $m, n > N \Rightarrow |a_m - a_n| < 1$. (Here, 1 plays the role of ε.)

Thus, for all $n > N$ we have: $|a_n - a_{N+1}| < 1$

$\Rightarrow -1 < a_n - a_{N+1} < 1$

$\Rightarrow a_{N+1} - 1 < a_n < a_{N+1} + 1$

The point of this, is that all terms of the sequence after the N^{th} term are *bounded* by $a_{N+1} - 1$ and $a_{N+1} + 1$.

Hence, $L = \min(a_1, a_2, a_3, \ldots, a_N, a_{N+1} - 1)$ is a lower bound of $\{a_n\}_{n=1}^{\infty}$

and $U = \max(a_1, a_2, a_3, \ldots, a_N, a_{N+1} + 1)$ is an upper bound of $\{a_n\}$

Thus the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded.

By the Bolzano-Weierstrass Theorem, the sequence has a limit point. ■

3. Every Cauchy Sequence is convergent.

(i.e., Every Cauchy sequence converges to a limit, L.)

Proof. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Let $\varepsilon > 0$ be given.

Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $m, n > N \Rightarrow |a_m - a_n| < \underline{\frac{\varepsilon}{2}}$

Also, since the sequence is Cauchy, it has at least one limit point.

Let L be a limit point of $\{a_n\}_{n=1}^{\infty}$.

Since L is a limit point of $\{a_n\}_{n=1}^{\infty}$, every open interval of the form $\underline{\left(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\right)}$ contains infinitely many terms of the sequence.

Consequently, $\exists k > N$ such that $a_k \in \underline{\left(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\right)}$

This is equivalent to saying that $L - \frac{\varepsilon}{2} < a_k < L + \frac{\varepsilon}{2}$

$\Leftrightarrow -\frac{\varepsilon}{2} < a_k - L < \frac{\varepsilon}{2}$

$\Leftrightarrow |a_k - L| < \frac{\varepsilon}{2}$

i.e., $\exists k > \mathbf{N}$ such that $|a_k - L| < \frac{\varepsilon}{2}$.

Observe: $\forall n > N$, we have:

$|a_n - L| = |a_n - a_k + a_k - L| \leq |a_n - a_k| + |a_k - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

i.e., $n > N \Rightarrow |a_n - L| < \varepsilon$.

Hence, $\{a_n\}_{n=1}^{\infty}$ converges to L. ■

Remark 1 *Since the limit of a convergence sequence is unique, L is unique. Thus the theorem (problem 2) which tells us that: "Every Cauchy Sequence has a limit point," can be modified to say that: "Every Cauchy Sequence has exactly one limit point."*

Scratchwork: We want $|a_n - L| < \varepsilon$.

Somehow, we have to relate $|a_n - L|$ to the fact that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Since L is a limit of the sequence $\{a_n\}_{n=1}^{\infty}$, every open interval of the form $(L-\delta, L+\delta)$ contains infinitely many terms of the sequence. Thus, there are infinitely many terms of the sequence within δ units of L (i.e., $|a_n - L| < \delta$ for infinitely many terms a_n of the seuqence.) However, this fact alone doesn't guarantee that $|a_n - L| < \delta$ for ALL terms a_n for which $n > N$ for some $N \in \mathbf{N}$.

Thus, for some k for which $|a_k - L| < \delta$, we must find a way to relate "ALL" terms of the sequence to a_k, and thus to L.

How do we do this?

Recall:

Since $\{a_n\}_{n=1}^{\infty}$ is Cauchy, given any $\delta > 0$, $\exists N$ such that $m, n > N \Rightarrow |a_m - a_n| < \delta$.

Having selected N, recall that we can find a $k > N$ such that $|a_k - L| < \delta$.

Thus, we have for ALL $n > N$:

$|a_n - L| = |a_n - a_k + a_k - L| \leq |a_n - a_k| + |a_k - L| < \delta + \delta = 2\delta$.

i.e., $|a_n - L| < 2\delta$

Since we want $|a_n - L| < \varepsilon$, we let $\varepsilon = 2\delta$ (or $\delta = \frac{\varepsilon}{2}$)

In retrospect, we require that $\exists N$ such that $m, n > N \Rightarrow |a_m - a_n| < \frac{\varepsilon}{2}$

and that every open interval of the form $\left(L - \frac{\varepsilon}{2}., L + \frac{\varepsilon}{2}\right)$ contains infinitely many terms of the sequence.

4. (Corollary) A sequence $\{a_n\}_{n=1}^{\infty}$ converges if and only if it is a Cauchy sequence.

Definition 2 *A function* $f : \mathbf{R} \rightarrow \mathbf{R}$ *is* ***continuous at a point*** x_0 *if*

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

Alternatively,a function $f : \mathbf{R} \rightarrow \mathbf{R}$ *is* ***continuous at a point*** x_0 *if*

$$\lim_{\Delta x \rightarrow 0} f(x_0 + \Delta x) = f(x_0).$$

Alternatively, a function $f : \mathbf{R} \rightarrow \mathbf{R}$ *is* ***continuous at a point*** x_0 *if*

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon, x_0) > 0 \text{ such that}$$

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Definition 3 *A function* $f : \mathbf{R} \rightarrow \mathbf{R}$ *is* ***continuous if it is continuous at each point*** $x_0 \in \mathbf{R}$.

Similarly, given that $\mathbf{B} \subseteq \mathbf{R}$*, a function* $f : \mathbf{B} \rightarrow \mathbf{R}$ *is* ***continuous if it is continuous at each point*** $x_0 \in \mathbf{B}$.

5. The constant function $f(x) = c$ is continuous.

   Proof. Let $f(x) = c$ and let x_0 and $\varepsilon > 0$ be given.
   Let $\delta = \delta(\varepsilon, x_0)$ be given by $\delta = 1$.

   Remark: As we will see, this choice of δ is completely arbitrary - any positive real number will work for our choice of δ.

   Observe: Given that $|x - x_0| < \delta$, we have:

   $|f(x) - f(x_0)| = |c - c| = 0 < \varepsilon$.

   i.e., $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

   Hence, the constant function $f(x) = c$ is continuous at all points $x_0 \in \mathbf{R}$. ■

6. The sum, difference, and product of continuous functions is continuous.

   Proof. Let $f(x), g(x)$ be continuous real-valued functions. Then:

   $\lim_{x\to x_0}(f(x)g(x)) = (\lim_{x\to x_0} f(x))(\lim_{x\to x_0} g(x))$ (lim of a product = prod of limits)

   $= f(x_0)g(x_0)$ (by continuity of $f(x)$ and $g(x)$).

   i.e., $\lim_{x\to x_0}(f(x)g(x)) = f(x_0)g(x_0)$

   Hence, the product of continuous functions is continuous at all points in their domain.

   (The proofs for sums and differences are analogous.) ■

7. The function $f(x) = x$ is continuous.

   Proof. Let $f(x) = x$ and let x_0 and $\varepsilon > 0$ be given.
   Let $\delta = \delta(\varepsilon, x_0)$ be given by $\delta = \varepsilon$.

   Remark: This choice of δ should become obvious in a second.

   Observe: Given that $|x - x_0| < \delta$, we have:

   $|f(x) - f(x_0)| = |x - x_0| < \delta = \varepsilon.$

   i.e., $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$

   Hence, the constant function $f(x) = x$ is continuous at all points $x_0 \in \mathbf{R}$. ■

8. The function $f(x) = x^n$ is continuous.

   Proof. Observe: $f(x) = x^n = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{n \text{ factors of } x}$ which is the product of continuous functions, by the preceding problem. Hence, $f(x) = x^n$ is continuous. ■

Definition 4 *A **monomial** in the variable x is an expression of the form: cx^n, where $c \in \mathbf{R}$ and n is a non-negative integer.*

Definition 5 *A **polynomial** in the variable x is a sum or difference of monomials in the variable x.*

9. All monomials are continuous.

   Proof. Observe: Let cx^n be any monomial. Then it is a product of the contious functions c and x^n. ■

10. All polynomials are continuous.

    Proof. This is a consequence of the fact that any polynomial is the sum of monomials and is, therefore, the sum of continuous functions. Hence, it is continuous. ■

11. Suppose that $f(x)$ and $g(x)$ are defined for all $x \in [a,b]$, except possibly $x = c$. Suppose also that $f(x) \leq g(x)$ for all $x \in [a,b]$, except possibly $x = c$. Then $\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$.

    Proof. Let the hypotheses be given and suppose that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$.

    Let $\varepsilon > 0$ be given.

    Let $\delta_1 = \delta_1(\varepsilon) > 0$ be given such that $|x - c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$.

    Let $\delta_2 = \delta_2(\varepsilon) > 0$ be given such that $|x - c| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$.

    Let $\delta = \min(\delta_1, \delta_2)$

    Then given $|x - c| < \delta$, we have:

    $|f(x) - L| < \frac{\varepsilon}{2}$ and $|g(x) - M| < \frac{\varepsilon}{2}$

    $\Rightarrow L - \frac{\varepsilon}{2} < f(x) < L + \frac{\varepsilon}{2}$ and $M - \frac{\varepsilon}{2} < g(x) < M + \frac{\varepsilon}{2}$

    $\Rightarrow L - \frac{\varepsilon}{2} < f(x) \leq g(x) < M + \frac{\varepsilon}{2}$

    $\Rightarrow L - \frac{\varepsilon}{2} < M + \frac{\varepsilon}{2}$

    $\Rightarrow L < M + \varepsilon$

    i.e., $\forall \varepsilon > 0,\ L < M + \varepsilon$.

    Hence, $L \leq M$

    i.e., $\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$. ■

12. (Sandwich Theorem) Suppose that $f(x), g(x)$ and $h(x)$ are defined for all $x \in [a,b]$, except possibly $x = c$. Suppose also that $f(x) \leq g(x) \leq h(x)$ for all $x \in [a,b]$, except possibly $x = c$. Finally, suppose that $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$. Then $\lim_{x \to c} g(x) = L$ also.

Proof. By the previous lemma, $\underbrace{\lim_{x \to c} f(x)}_{=L} \leq \lim_{x \to c} g(x)$.

Also, $\lim_{x \to c} g(x) \leq \underbrace{\lim_{x \to c} h(x)}_{=L}$

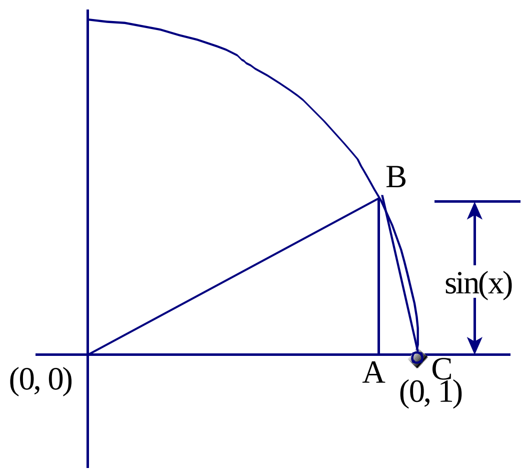
Thus, $L \leq \lim_{x \to c} g(x) \leq L$

$\Rightarrow \lim_{x \to c} g(x) = L$ ■

13. $\lim_{x \to 0} \sin(x) = 0$

Proof. We will show that $\lim_{x \to 0^+} \sin(x) = 0$. The proof for $\lim_{x \to 0-} \sin(x)$ is similar.

Observe that for $x \in \left[0, \frac{\pi}{2}\right]$, we have:



$0 \leq \overline{AB} = \sin(x) \leq \overline{BC} \leq arc(BC) = x$

i.e., $0 \leq \sin(x) \leq x$

Hence: $\underbrace{\lim_{x \to 0} 0}_{=0, \text{ (limit of a constant)}} \leq \lim_{x \to 0} \sin(x) \leq \underbrace{\lim_{x \to 0} x}_{=0, \text{ by continuity of } x}$

i.e., $0 \leq \lim_{x \to 0} \sin(x) \leq 0$

Hence, $\lim_{x \to 0} \sin(x) = 0$ ■

14. $\lim_{x\to 0}\cos(x) = 1$

Proof. We will show that $\lim_{x\to 0^+}\cos(x) = 1$. The proof for $\lim_{x\to 0-}\cos(x)$ is similar.

Let $\varepsilon > 0$ be given. WOLOG, $\varepsilon < 1$

Also, WOLOG, since $x \to 0^+$, we assume that $0 < x < \frac{\pi}{2}$

Since $\lim_{x\to 0^+}\sin(x) = 0$, it follows that $\lim_{x\to 0^+}\sin^2(x) = 0$ also. (limit of a product ...)

Hence, $\exists\delta = \delta(\varepsilon) > 0$ such that $0 < |x - 0| < \delta$

$\Rightarrow \left|\sin^2(x) - 0\right| < \varepsilon$

Consequently, $0 < x < \delta \Rightarrow \sin^2(x) < \varepsilon$.

Observe: $1 - \varepsilon < \sqrt{1-\varepsilon} < \sqrt{1-\sin^2(x)} = \underbrace{\cos(x) < 1}_{\text{because } x \gneqq 0}$

i.e., $1 - \varepsilon < \cos(x) < 1$

$\Rightarrow 1 - \varepsilon < \cos(x) < 1 + \varepsilon$

$\Rightarrow -\varepsilon < \cos(x) - 1 < \varepsilon$

$\Rightarrow |\cos(x) - 1| < \varepsilon$

i.e., $0 < x < \delta \Rightarrow |\cos(x) - 1| < \varepsilon$

Hence, $\lim_{x\to 0^+}\cos(x) = 1$ ■

15. The function $f(x) = \sin(x)$ is continuous on the interval $(-\infty, \infty)$.

    Proof. $\lim_{x \to x_0} \sin(x) = \lim_{\Delta x \to 0} \sin(x_0 + \Delta x)$

    $= \lim_{\Delta x \to 0} [\sin(x_0)\cos(\Delta x) + \cos(x_0)\sin(\Delta x)]$

    $= \lim_{\Delta x \to 0} \sin(x_0)\cos(\Delta x) + \lim_{\Delta x \to 0} \cos(x_0)\sin(\Delta x)$

    $= \sin(x_0)(\lim_{\Delta x \to 0} \cos(\Delta x)) + \cos(x_0)(\lim_{\Delta x \to 0} \sin(\Delta x))$

    $= \sin(x_0) \cdot 1 + \cos(x_0) \cdot 0 = \sin(x_0)$

    i.e., $\lim_{x \to x_0} \sin(x) = \sin(x_0)$.

    Hence, $f(x) = \sin(x)$ is continuous at all points $x_0 \in \mathbf{R}$. ■

16. The function $f(x) = \cos(x)$ is continuous on the interval $(-\infty, \infty)$.

    Proof. Observe: $\lim_{x \to x_0} \cos(x) = \lim_{\Delta x \to 0} \cos(x_0 + \Delta x)$

    $= \lim_{\Delta x \to 0} [\cos(x_0)\cos(\Delta x) - \sin(x_0)\sin(\Delta x)]$

    $= \lim_{\Delta x \to 0} \cos(x_0)\cos(\Delta x) - \lim_{\Delta x \to 0} \sin(x_0)\sin(\Delta x)$

    $= \cos(x_0)(\lim_{\Delta x \to 0} \cos(\Delta x)) - \sin(x_0)(\lim_{\Delta x \to 0} \sin(\Delta x))$

    $= \cos(x_0) \cdot 1 - \sin(x_0) \cdot 0 = \cos(x_0)$

    i.e., $\lim_{x \to x_0} \cos(x) = \cos(x_0)$

    Hence, $f(x) = \sin(x)$ is continuous at all points $x_0 \in \mathbf{R}$. ■

17. The function e^x is continuous on the interval $(-\infty, \infty)$ and the function $\ln(x)$ is continuous on the interval $(0, \infty)$.

    We hold the proofs of these in abeyance.

18. The quotient $\frac{f(x)}{g(x)}$ of continuous functions, $f(x)$ and $g(x)$, is continuous at all points, except at those values of x, where the denominator equals 0.

    We hold the proofs of this in abeyance.

Remark: The following theorem relates sequences and continuity by giving an alternate characterization of "continuous at a point."

19. A real valued function $f(x)$ is continuous at a point $x = c$ if and only if $\{f(a_n)\}_{n=1}^{\infty}$ converges to $f(c)$ for each sequence $\{a_n\}_{n=1}^{\infty}$ that converges to c.

Proof.

$(f(x)$ is continuous at $x = c) \quad \Rightarrow \quad (f(a_n) \to f(c)\ \forall$ sequences $\{a_n\}_{n=1}^{\infty}$ that converge to $c.)$

Let the hypothesis be given. (i.e., suppose that $f(x)$ is continuous at $x = c$.)

Let $\varepsilon > 0$ be given.

Since $f(x)$ is continuous at $x = c$, $\exists\ \delta = \delta(\varepsilon) > 0$, such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.

Suppose that $\{a_n\}_{n=1}^{\infty}$ converges to c.

Then $\exists\ N = N(\delta) \in \mathbf{N}$, such that $n > N \Rightarrow |a_n - c| < \delta$.

Observe: Given $n > N$, we have:

$|a_n - c| < \delta \Rightarrow |f(a_n) - f(c)| < \varepsilon$.

i.e., $n > N \Rightarrow |f(a_n) - f(c)| < \varepsilon$.

Hence, $\{f(a_n)\}_{n=1}^{\infty}$ converges to $f(c)$.

$(f(a_n) \to f(c)\ \forall$ sequences $\{a_n\}_{n=1}^{\infty}$ that converge to $c.) \quad \Rightarrow \quad (f(x)$ is continuous at $x = c)$

Let the hypothesis be given. (i.e., suppose that $f(a_n) \to f(c)\ \forall$ sequences $\{a_n\}_{n=1}^{\infty}$ that converge to L.)

Suppose, for the sake of deriving a contradiction, that $f(x)$ is NOT continuous at $x = c$.

Then $\exists\ \varepsilon > 0$, such that $\forall \delta > 0$, $\exists\ x$ such that:

$|x - c| < \delta$, and yet $|f(x) - f(c)| > \varepsilon$.

Given such an ε, let $\delta = 1$.

Then $\exists\ x$ (call it x_1) such that $|x_1 - c| < 1$, and yet $|f(x_1) - f(c)| > \varepsilon$.

For the same ε, let $\delta = \frac{1}{2}$.

Then $\exists\ x$ (call it x_2) such that $|x_2 - c| < \frac{1}{2}$, and yet $|f(x_2) - f(c)| > \varepsilon$.

Continuing inductively, we obtain a sequence $\{x_n\}_{n=1}^{\infty}$ such that $|x_n - c| < \frac{1}{n}\ \forall n \in \mathbf{N}$ (i.e., $\{x_n\}_{n=1}^{\infty}$ converges to c), and yet the interval $(f(c) - \varepsilon, f(c) + \varepsilon)$ contains no terms of the sequence $\{f(x_n)\}_{n=1}^{\infty}$.

(i.e., $x_n \to c$, and yet $f(x_n) \not\to f(c)$.)

This contradicts our hypothesis.

Since the assumption that $f(x)$ is not continuous at $x = c$ leads to a contradiction, the assumption must be false.

Hence, $f(x)$ is continuous at the point $x = c$. ■

20. State and prove the Heine - Borel Theorem

Let $G = \{(\alpha_i, \beta_i)\}_{i \in I}$ be an open cover of $[a, b]$. Then G contains a finite sub-covering of $[a, b]$. (i.e., $\exists \left(\alpha_{i_1}, \beta_{i_1}\right), \left(\alpha_{i_2}, \beta_{i_2}\right), \ldots, \left(\alpha_{i_k}, \beta_{i_k}\right) \in G$, such that $[a, b] \subseteq \cup_{j=1}^{k} \left(\alpha_{i_j}, \beta_{i_j}\right)$.)

Proof. Suppose, for the sake of deriving a contradiction, that no finite sub-cover exists.

Partition $[a, b]$ into two subintervals, $[a, c]$ and $[c, b]$, where c is the midpoint of $[a, b]$. At least one of these subintervals cannot be covered by finitely many subintervals of G (otherwise, both $[a, c]$ and $[c, b]$ can be covered by finitely many intervals of G, and hence $[a, b]$ can be covered by finitely many intervals of G (because $[a, b] = [a, c] \cup [c, b]$), contrary to our assumption).

Select one of the subintervals which cannot be covered by finitely many intervals of G, and call it $[a_1, b_1]$.

Partition $[a_1, b_1]$ into two subintervals, $[a_1, c_1]$ and $[c_1, b_1]$, where c_1 is the midpoint of $[a_1, b_1]$. At least one of these subintervals cannot be covered by finitely many subintervals of G.

Select one of the subintervals which cannot be covered by finitely many intervals of G, and call it $[a_2, b_2]$.

Proceeding inductively, we create a nested sequence of intervals:

$$[a, b] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \ldots \supseteq [a_n, b_n] \supseteq \ldots$$

with the property that

$$\lim_{n \to \infty} (b_n - a_n) = 0.$$

By the Nested Interval Theorem, $\exists!\ p \in \mathbf{R}$ such that

$$\cap_{n=1}^{\infty} [a_n, b_n] = p.$$

Since $p \in [a_n, b_n]\ \forall n \in \mathbf{N}$, and since G covers $[a_n, b_n]\ \forall n \in \mathbf{N}$, $\exists (\alpha_i, \beta_i) \in G$ such that $p \in (\alpha_i, \beta_i)$.

Let $\varepsilon = \min [(\beta_i - p), (p - \alpha_i)]$.

Since $\lim_{n \to \infty} (b_n - a_n) = 0$, $\exists N \in \mathbf{N}$ such that $n > N \Rightarrow (b_n - a_n) < \varepsilon$.

Given such an n, it follows that $p \in [a_n, b_n] \subseteq (\alpha_i, \beta_i)$.

The fact that $[a_n, b_n] \subseteq (\alpha_i, \beta_i)$ contradicts the fact that $[a_n, b_n]$ cannot be covered by finitely many intervals of G.

Since the assumption that $[a, b]$ cannot be covered by finitely many intervals of G led to this contradiction, this assumption must be false.

Hence, $[a, b]$ can be covered by finitely many intervals of G. ■

21. If $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$, then the *image* of $[a, b]$, $\{f(x) : x \in [a, b]\}$, is bounded.

Proof. Let the hypothesis be given. (Note that $[a, b]$ is closed and finite.)

Remark: Our goal is to get an open cover of the *image* of $[a, b]$, $\{f(x) : x \in [a, b]\}$, that can be reduced to a finite subcover. Since we don't know whether or not the *image* of $[a, b]$ is a closed interval of finite length, this may not be easy.

For each point $f(x)$ in the image of $[a, b]$, consider the open interval $(f(x) - \varepsilon, f(x) + \varepsilon)$. Since f is continuous at each x_0 in the interval $[a, b]$, there exists, for each $x_0 \in [a, b]$, a $\delta_{x_0} = \delta_{x_0}(\varepsilon) > 0$ such that $|x - x_0| < \delta_{x_0} \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

i.e., $x \in (x_0 - \delta_{x_0}, x_0 + \delta_{x_0}) \Rightarrow f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

Notice that the set of open intervals $G = \{(x - \delta_x, x + \delta_x)\}_{x \in [a,b]}$ is an open cover of the interval $[a, b]$.

By the Heine-Borel Theorem, G can be reduced to a finite sub-cover of $[a, b]$.

Let $\{(x_i - \delta_{xi}, x_i + \delta_{xi})\}_{i=1}^{n}$ be such a finite sub-cover of $[a, b]$.

Since every $x \in [a, b]$, is contained in at least one of the intervals,
$(x_1 - \delta_{x_1}, x_1 + \delta_{x_1}), (x_2 - \delta_{x_2}, x_2 + \delta_{x_2}), \ldots, (x_n - \delta_{x_n}, x_n + \delta_{x_n})$,

every point $f(x)$ in the *image* of $[a, b]$ must be contained in one of the intervals

$(f(x_1) - \varepsilon, f(x_1) + \varepsilon), (f(x_2) - \varepsilon, f(x_2) + \varepsilon), \ldots, (f(x_n) - \varepsilon, f(x_n) + \varepsilon)$.

Hence, every point $f(x)$ in the *image* of $[a, b]$ must be such that:

$$\min(f(x_1) - \varepsilon, f(x_2) - \varepsilon, \ldots, f(x_n) - \varepsilon) \le f(x) \le \max(f(x_1) + \varepsilon, f(x_2) + \varepsilon, \ldots, f(x_n) + \varepsilon)$$

Thus, $\min(f(x_1) - \varepsilon, f(x_2) - \varepsilon, \ldots, f(x_n) - \varepsilon)$ is a lower bound of the *image* of $[a, b]$, and $\max(f(x_1) + \varepsilon, f(x_2) + \varepsilon, \ldots, f(x_n) + \varepsilon)$ is an upper bound. ■

22. State and prove the Extreme Value Theorem.

Suppose that $f(x) : [a, b] \to \mathbf{R}$ is continuous. Then $f(x)$ attains both an absolute maximum value and an absolute minimum value. (Notice that the domain is a *closed* interval of *finite* length.)

Proof. We shall prove the case for $f(x)$ having an absolute maximum. The case for the absolute minimum is similar.

We have already proved that if $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$, then the *image* of $[a, b]$, (i.e. $\{f(x) : x \in [a, b]\}$) is bounded.

Since the image of $f(x)$ is bounded, it must have a least upper bound. We'll call it M.

Remark: By definition of *least upper bound,* $M \geq f(x), \forall x \in [a, b]$. This is not quite the same thing as saying that $\exists x \in [a, b]$ such that $f(x) = M$. Our challenge in this proof is to prove that such an x exists. (End of Remark)

Partition the interval $[a, b]$ into two subintervals, $[a, c]$ and $[c, b]$, where c is the midpoint of the original interval.

Observe: M is an upper bound of the *images* of both subintervals, and is the *least* upper bound of at least one of the images. (i.e. M is the l.u.b. of either $\{f(x) : x \in [a, c]\}$ or $\{f(x) : x \in [c, b]\}$.)

Choose one of the subintervals, of whose image M is the least upper bound, and call it $[a_1, b_1]$. Partition the interval into two subintervals, $[a_1, c_1]$ and $[c_1, b_1]$, where c_1 is the midpoint of $[a_1, b_1]$. Again, M is an upper bound of the *images* both subintervals, and is the *least* upper bound of at least one of the images. (i.e. M is the l.u.b. of either $\{f(x) : x \in [a_1, c_1]\}$ or $\{f(x) : x \in [c_1, b_1]\}$.) We choose one of the subintervals, of whose image M is the least upper bound, and call it $[a_2, b_2]$.

We continue the process inductively, and in so doing, we create a sequence of nested intervals:

$$[a, b] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \ldots \supseteq [a_n, b_n] \supseteq \ldots$$

with the property that

$$\lim_{n \to \infty} (b_n - a_n) = 0.$$

Observe, for future reference, that M is the least upper bound of the *image* of each interval $[a_n, b_n]$.

By the *Nested Interval Theorem,* there exists a single real number p such that

$$\cap_{n=1}^{\infty} [a_n, b_n] = p.$$

We claim that $f(p) = M$. (Clearly, $f(p) \leq M$, since $p \in [a_n, b_n]\ \forall n \in \mathbf{N}$, and since M is the least upper bound of the image $\{f(x) : x \in [a_n, b_n]\}$ of each $[a_n, b_n]$.)

So, if we suppose, for the sake of deriving a contradiction, that $f(p) \neq M$, then it must be the case that $f(p) \lneqq M$.

If $f(p) \lneqq M$, then $\exists \varepsilon > 0$, such that $M - f(p) = \varepsilon$.

Recall that f is continuous at each point in the interval $[a, b]$, (and continuous, in particular, at the point p).

Consequently, $\exists \delta = \delta(\varepsilon) > 0$, such that $|x - p| < \delta \Rightarrow |f(x) - f(p)| < \frac{\varepsilon}{2}$.

Also, since $\lim_{n \to \infty} (b_n - a_n) = 0$, $\exists N \in \mathbf{N}$ such that $n > N \Rightarrow (b_n - a_n) < \delta$.

Given such an n, it follows that $|x - p| < \delta$, $\forall x \in [a_n, b_n]$.

Hence, $\forall x \in [a_n, b_n]$, $|f(x) - f(p)| < \frac{\varepsilon}{2}$.

$\Rightarrow -\frac{\varepsilon}{2} < f(x) - f(p) < \frac{\varepsilon}{2}$

$\Rightarrow f(p) - \frac{\varepsilon}{2} < f(x) < f(p) + \frac{\varepsilon}{2}$.

i.e., $\forall x \in [a_n, b_n]$, $f(x) < f(p) + \frac{\varepsilon}{2} < f(p) + \varepsilon = M$.

Since $f(p) + \frac{\varepsilon}{2}$ is an upper bound of $\{f(x) : x \in [a_n, b_n]\}$ that is less than M, this contradicts the fact that M is the least upper bound of $\{f(x) : x \in [a_n, b_n]\}$.

Since the assumption that $f(p) \neq M$ led to the contradiction, it must be the case that $f(p) = M$.

i.e., $\exists x \in [a, b]$, namely p, such that $f(x) = M$. Hence, $f(x)$ attains an absolute maximum value on $[a, b]$. ■

Definition 6 *Let $f : \mathbf{R} \to \mathbf{R}$. The* ***derivative*** *of $f(x)$ at the point $(x_0, f(x_0))$, denoted $f'(x_0)$, is given by:*

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Alternatively: *Let $f : \mathbf{R} \to \mathbf{R}$. The* ***derivative*** *of $f(x)$ at the point $(x_0, f(x_0))$, denoted $f'(x_0)$, is given by:*

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Definition 7 *If $f'(x_0)$ exists, then we say that $f(x)$ is* ***differentiable at the point*** *x_0. If $f(x)$ is differentiable $\forall x_0 \in \mathbf{R}$, then we say that $f(x)$ is* ***differentiable.***

23. **Prove:** Differentiability implies continuity.

    Proof. Let $f(x)$ be differentiable at x_0. Then $f'(x_0)$ exists, and

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

In order to show that $f(x)$ is continuous at x_0, we must show that

$$\lim_{x \to x_0} f(x) = f(x_0)$$

or equivalently (letting $x = x_0 + \Delta x$),

$$\lim_{\Delta x \to 0} f(x_0 + \Delta x) = f(x_0).$$

Observe:

$$\begin{aligned}
\lim_{\Delta x \to 0} f(x_0 + \Delta x) &= \lim_{\Delta x \to 0} [f(x_0) + f(x_0 + \Delta x) - f(x_0)] \\
&= \lim_{\Delta x \to 0} [f(x_0)] + \lim_{\Delta x \to 0} [f(x_0 + \Delta x) - f(x_0)] \\
&= f(x_0) + \lim_{\Delta x \to 0} [f(x_0 + \Delta x) - f(x_0)] \\
&= f(x_0) + \lim_{\Delta x \to 0} \Delta x \tfrac{[f(x_0 + \Delta x) - f(x_0)]}{\Delta x} \\
&= f(x_0) + (\lim_{\Delta x \to 0} \Delta x) \cdot \left(\lim_{\Delta x \to 0} \tfrac{[f(x_0 + \Delta x) - f(x_0)]}{\Delta x}\right) \\
&= f(x_0) + 0 \cdot f'(x_0) \\
&= f(x_0)
\end{aligned}$$

i.e., $\lim_{\Delta x \to 0} f(x_0 + \Delta x) = f(x_0)$.

Hence, $f(x)$ is continuous at $x = x_0$. ■

Remark 2 *By definition, a function $f(x)$ is* continuous at a point x_0, *if*

$$\lim_{x \to x_0} f(x) = f(x_0)$$

or equivalently, if

$$\lim_{\Delta x \to x_0} f(x_0 + \Delta x) = f(x_0).$$

Since differentiability implies continuity, we can assume from here on, that if $f(x)$ is differentiable at x_0, then

$$\lim_{x \to x_0} f(x) = f(x_0)$$

or equivalently,

$$\lim_{\Delta x \to x_0} f(x_0 + \Delta x) = f(x_0).$$

Remark 3 *The symbol $\frac{d}{dx}$ is a differential operator which instructs us to compute the derivative of the expression that follows. We interpret the expression $\frac{d}{dx}[f(x)]$ to be the derivative of $f(x)$.*

$$\text{i.e., } \frac{d}{dx}[f(x)] \text{ is the same as } f'(x).$$

24. **Prove:** $\frac{d}{dx}[c] = 0$; where c is a constant

    Proof. Let $h(x) = c$

    Observe:

    $\frac{d}{dx}[c] = h'(x) = \lim_{\Delta x \to 0} \frac{h(x_0+\Delta x)-h(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c-c}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0$

    i.e., $\frac{d}{dx}[c] = 0$. ■

25. $\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$; (where c is a constant and $f(x)$ is differentiable.)

    Proof. Let $h(x) = c \cdot f(x)$

    Observe:

    $\frac{d}{dx}[c \cdot f(x)] = h'(x) = \lim_{\Delta x \to 0} \frac{h(x_0+\Delta x)-h(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c \cdot f(x+\Delta x) - c \cdot f(x)}{\Delta x}$

    $= \lim_{\Delta x \to 0} c \cdot \frac{f(x+\Delta x)-f(x)}{\Delta x} = c \cdot \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = c \cdot f'(x)$

    i.e., $\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$ ■

26. $\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$ (provided that $f(x), g(x)$ are differentiable.)

Proof. We will prove that $\frac{d}{dx}\left[f(x) + g(x)\right] = f'(x) + g'(x)$.

The proof that $\frac{d}{dx}\left[f(x) - g(x)\right] = f'(x) - g'(x)$ is analogous.

Let $h(x) = f(x) + g(x)$.

Observe:

$\frac{d}{dx}\left[f(x) + g(x)\right] = h'(x) = \lim_{\Delta x\to 0} \frac{h(x+\Delta x)-h(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{[f(x+\Delta x)+g(x+\Delta x)]-[f(x)+g(x)]}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{[f(x+\Delta x)-f(x)]+[g(x+\Delta x)-g(x)]}{\Delta x} = \lim_{\Delta x\to 0} \left[\frac{[f(x+\Delta x)-f(x)]}{\Delta x} + \frac{[g(x+\Delta x)-g(x)]}{\Delta x}\right]$

$= \lim_{\Delta x\to 0} \left[\frac{[f(x+\Delta x)-f(x)]}{\Delta x}\right] + \lim_{\Delta x\to 0} \left[\frac{[g(x+\Delta x)-g(x)]}{\Delta x}\right] = f'(x) + g'(x)$

i.e., $\frac{d}{dx}\left[f(x) + g(x)\right] = f'(x) + g'(x)$ ■

27. $\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + g'(x) \cdot f(x)$ (provided that $f(x), g(x)$ are differentiable.)

Proof. Let $h(x) = f(x) \cdot g(x)$

Observe:

$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = h'(x) = \lim_{\Delta x\to 0} \frac{h(x+\Delta x)-h(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)\cdot g(x+\Delta x)-\overbrace{\left[f(x)\cdot g(x+\Delta x) - f(x)\cdot g(x+\Delta x)\right]}^{=0}-f(x)\cdot g(x)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{[f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x+\Delta x)]+[f(x)\cdot g(x+\Delta x)-f(x)\cdot g(x)]}{\Delta x}$

$= \lim_{\Delta x\to 0} \left[\frac{f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x+\Delta x)}{\Delta x} + \frac{f(x)\cdot g(x+\Delta x)-f(x)\cdot g(x)}{\Delta x}\right]$

$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x+\Delta x)}{\Delta x} + \lim_{\Delta x\to 0} \frac{f(x)\cdot g(x+\Delta x)-f(x)\cdot g(x)}{\Delta x}$

$= \lim_{\Delta x\to 0} g(x+\Delta x) \frac{f(x+\Delta x)-f(x)}{\Delta x} + \lim_{\Delta x\to 0} f(x) \frac{g(x+\Delta x)-g(x)}{\Delta x}$

$= \lim_{\Delta x\to 0} g(x+\Delta x) \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} + \underbrace{f(x)}_{\substack{f(x) \text{ is a} \\ \text{constant w.r.t } \Delta x}} \lim_{\Delta x\to 0} \frac{g(x+\Delta x)-g(x)}{\Delta x}$

$= \underbrace{g(x)}_{\substack{\lim_{\Delta x\to 0} g(x+\Delta x)=g(x) \\ \text{by continuity of } g(x)}} \cdot f'(x) + \underbrace{f(x)}_{\substack{f(x) \text{ is a} \\ \text{constant w.r.t } \Delta x}} \cdot g'(x)$

$= f'(x) \cdot g(x) + g'(x) \cdot f(x)$

i.e. $\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + g'(x) \cdot f(x)$ ■

28. $\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{[g(x)]^2}$; provided that $g(x) \neq 0$ is differentiable.

Proof. Observe: $0 = \frac{d}{dx}[1] = \frac{d}{dx}\left[g(x) \cdot \frac{1}{g(x)}\right] = \underbrace{g'(x) \cdot \frac{1}{g(x)} + \left(\frac{d}{dx}\left[\frac{1}{g(x)}\right]\right) \cdot g(x)}_{\text{By the Product Rule}}$

i.e., $g'(x) \cdot \frac{1}{g(x)} + \underbrace{\left(\frac{d}{dx}\left[\frac{1}{g(x)}\right]\right)}_{\text{This is what we want}} \cdot g(x) = 0$

$\Rightarrow \left(\frac{d}{dx}\left[\frac{1}{g(x)}\right]\right) \cdot g(x) = -g'(x) \cdot \frac{1}{g(x)}$

$\Rightarrow \left(\frac{d}{dx}\left[\frac{1}{g(x)}\right]\right) = -\frac{g'(x)}{[g(x)]^2}$ ■

29. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2}$; (provided that $f(x), g(x)$ are differentiable and $g(x) \neq 0$.)

Proof. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left[f(x) \cdot \frac{1}{g(x)}\right] = \underbrace{f'(x) \cdot \frac{1}{g(x)} + \left(-\frac{g'(x)}{[g(x)]^2}\right) \cdot f(x)}_{\text{By the Product Rule}}$

$= \frac{f'(x)}{g(x)} - \frac{g'(x)f(x)}{[g(x)]^2} = \frac{f'(x)g(x)}{[g(x)]^2} - \frac{g'(x)f(x)}{[g(x)]^2} = \frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2}$

i.e., $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2}$ ■

30. **Prove** Pascal's Rule: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $1 \leq k \leq n$

Proof. Observe: $\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)![n-(k-1)]!} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)![(n-k)+1]!} =$
$\frac{n!}{k!(n-k)!} \cdot \frac{[(n-k)+1]}{[(n-k)+1]} + \frac{n!}{(k-1)![(n-k)+1]!} \cdot \frac{k}{k} = \frac{n![(n-k)+1]+n!k}{k![(n-k)+1]!} = \frac{n![(n+1)-k]+n!k}{k![(n+1)-k]!} = \frac{n!(n+1)-n!k+n!k}{k![(n+1)-k]!} =$
$\frac{(n+1)!}{k![(n+1)-k]!} = \binom{n+1}{k}$

i.e., $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ ■

31. **Prove** the Binomial Theorem: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$ $\forall a, b \in \mathbf{R}$ and $\forall n \in \mathbf{N} \cup \{0\}$

Proof. (by induction on n)

Show true for $n = 0$

$(a+b)^0 = 1 = 1 \cdot a^0 b^{0-0} = \sum_{i=0}^{0} \binom{0}{i} a^i b^{0-i}$

i.e., $(a+b)^0 = \sum_{i=0}^{0} \binom{0}{i} a^i b^{0-i}$

Assume true for $n = k$, and show true for $n = k+1$.

i.e., assume that $(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i}$, and use this assumption to show that

$$(a+b)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} a^i b^{(k+1)-i}$$

Observe: $(a+b)^{k+1} = (a+b)^k (a+b) = \left(\sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i}\right)(a+b) =$

$$\left(\sum_{i=0}^{k} \binom{k}{i} a^{i+1} b^{k-i}\right) + \left(\sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i+1}\right) = \underbrace{\left(\sum_{j=1}^{k+1} \binom{k}{j-1} a^j b^{k-j+1}\right)}_{\text{let } j=i+1} + \left(\sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i+1}\right)$$

$$= \underbrace{\left(\sum_{i=1}^{k+1} \binom{k}{i-1} a^i b^{k-i+1}\right)}_{\text{let } i=j} + \left(\sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i+1}\right)$$

$$= \left[\underbrace{a^{k+1} b^0}_{i=(k+1) \text{ term}} + \left(\sum_{i=1}^{k} \binom{k}{i-1} a^i b^{k-i+1}\right)\right] + \left[\left(\sum_{i=1}^{k} \binom{k}{i} a^i b^{k-i+1}\right) + \underbrace{a^0 b^{k+1}}_{i=0 \text{ term}}\right]$$

$= a^0 b^{k+1} + \left(\sum_{i=1}^{k} \left[\binom{k}{i} + \binom{k}{i-1}\right] a^i b^{k-i+1}\right) + a^{k+1} b^0$

$$= a^0 b^{k+1} + \underbrace{\left(\sum_{i=1}^{k} \binom{k+1}{i} a^i b^{k-i+1}\right)}_{\text{by Pascal's Rule}} + a^{k+1} b^0$$

$= \binom{k+1}{0} a^0 b^{k+1} + \left(\sum_{i=1}^{k} \binom{k+1}{i} a^i b^{(k+1)-i}\right) + \binom{k+1}{k+1} a^{k+1} b^0$

$= \sum_{i=0}^{k+1} \binom{k+1}{i} a^i b^{(k+1)-i}$

i.e., $(a+b)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} a^i b^{(k+1)-i}$

Hence, $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$ $\forall a, b \in \mathbf{R}$ and $\forall n \in \mathbf{N} \cup \{0\}$ ■

32. **Prove:** $\frac{d}{dx}\left[x^n\right] = nx^{n-1} \quad \forall n \in \mathbf{N}$

Proof.

$$\begin{aligned}
\frac{d}{dx}\left[x^n\right] &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\left(\sum_{i=0}^{n} \binom{n}{i} x^i \Delta x^{n-i}\right) - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\left(x^n + nx^{n-1}\Delta x + \frac{n(n-1)x^{n-2}\Delta x^2}{2} + \ldots + \frac{n(n-1)x^2\Delta x^{n-2}}{2} + nx^1\Delta x^{n-1} + \Delta x^n\right) - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + \frac{n(n-1)x^{n-2}\Delta x^2}{2} + \ldots + \frac{n(n-1)x^2\Delta x^{n-2}}{2} + nx^1\Delta x^{n-1} + \Delta x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x\left(nx^{n-1} + \frac{n(n-1)x^{n-2}\Delta x}{2} + \ldots + \frac{n(n-1)x^2\Delta x^{n-2}}{2} + nx^1\Delta x^{n-2} + \Delta x^{n-1}\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(nx^{n-1} + \frac{n(n-1)x^{n-2}\Delta x}{2} + \ldots + \frac{n(n-1)x^2\Delta x^{n-2}}{2} + nx^1\Delta x^{n-2} + \Delta x^{n-1}\right) \\
&= nx^{n-1}
\end{aligned}$$

■

33. **Prove:** $\frac{d}{dx}\left[x^n\right] = nx^{n-1} \quad \forall n \in \mathbf{N}$ (prove by induction on n.)

Proof. **Step 1** Show true for $n = 1$

$\frac{d}{dx}\left[x^1\right] = \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1 = 1 \cdot x^{1-1}$

i.e., $\frac{d}{dx}\left[x^1\right] = 1 \cdot x^{1-1}$

Step 2 Assume true for $n = k$, and show true for $n = k+1$.

i.e., Assume that $\frac{d}{dx}\left[x^k\right] = kx^{k-1}$ and show that $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^k$

Observe:

$$\frac{d}{dx}\left[x^{k+1}\right] = \frac{d}{dx}\left[x \cdot x^k\right] = \underbrace{1 \cdot x^k + \left(\frac{d}{dx}\left[x^k\right]\right) \cdot x}_{\text{By the Product Rule}} = x^k + \underbrace{kx^{k-1}}_{\text{by Ind. Hyp.}} \cdot x = (k+1)\,x^k$$

i.e, $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^k$

Hence, $\frac{d}{dx}\left[x^n\right] = nx^{n-1} \quad \forall n \in \mathbf{N}$ ■

34. **Prove:** $\frac{d}{dx}\left[x^n\right] = nx^{n-1} \quad \forall n \in \mathbf{Z}$

Proof. Note that our proposition holds, trivially, for $n = 0$.

So suppose that k is a negative integer. Then $k = -n$ for some $n \in \mathbf{N}$.

Observe:

$$\frac{d}{dx}\left[x^k\right] = \frac{d}{dx}\left[x^{-n}\right] = \underbrace{\frac{d}{dx}\left[\frac{1}{x^n}\right] = -\frac{nx^{n-1}}{\left[x^n\right]^2}}_{\text{By the Reciprocal Rule}} = -n\frac{x^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1} = kx^{k-1}$$

i.e., $\frac{d}{dx}\left[x^k\right] = kx^{k-1}$ for all negative integers, k. ■

35. **Prove:** The power rule holds for all rational exponents. (i.e., $\frac{d}{dx}\left[x^{\frac{p}{q}}\right] = \frac{p}{q}x^{\frac{p}{q}-1}$, $\forall p, q \in \mathbf{Z}, q \neq 0$.)

Proof. Observe: $px^{p-1} = \frac{d}{dx}\left[x^p\right] = \frac{d}{dx}\left[\left(x^{\frac{p}{q}}\right)^q\right] = q\left(x^{\frac{p}{q}}\right)^{q-1} \cdot \underbrace{\left(\frac{d}{dx}\left[x^{\frac{p}{q}}\right]\right)}_{\text{This is what we want}}$

i.e., $q\left(x^{\frac{p}{q}}\right)^{q-1} \cdot \left(\frac{d}{dx}\left[x^{\frac{p}{q}}\right]\right) = px^{p-1}$

$\Rightarrow \frac{d}{dx}\left[x^{\frac{p}{q}}\right] = \frac{px^{p-1}}{q\left(x^{\frac{p}{q}}\right)^{q-1}} = \frac{px^{p-1}}{q\left(x^{\frac{pq-p}{q}}\right)} = \frac{px^{p-1}}{q\left(x^{p-\frac{p}{q}}\right)} = \frac{p}{q}x^{p-1-p+\frac{p}{q}} = \frac{p}{q}x^{\frac{p}{q}-1}$

i.e., $\frac{d}{dx}\left[x^{\frac{p}{q}}\right] = \frac{p}{q}x^{\frac{p}{q}-1}$ ■

36. Let $f(x)$ be differentiable on the interval $[a, b]$, and suppose that $f(x)$ attains a relative maximum or a relative minimum at a point $x_0 \in (a, b)$. Then $f'(x_0) = 0$.

Proof. Without loss of generality, we'll assume that $f(x)$ attains a relative maximum at x_0. (The case in which $f(x)$ attains a relative minimum is similar.)

Observe: $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$ for $x < x_0$

$\frac{f(x)-f(x_0)}{x-x_0} \leq 0$ for $x > x_0$

i.e., $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \geq 0$ for $\Delta x < 0$

$\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \leq 0$ for $\Delta x > 0$

Thus, $0 \leq \lim_{\Delta x \to 0^-} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = f'(x_0) = \lim_{\Delta x \to 0^+} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \leq 0$

i.e., $0 \leq f'(x_0) \leq 0$

$\Rightarrow f'(x_0) = 0$ ■

Corollary 8 *(Contrapositive of previous lemma 36) Suppose that $f(x)$ attains a relative maximum or a relative minimum at a point $x_0 \in (a, b)$ and that $f'(x_0) \neq 0$. Then $f'(x_0)$ does not exist.*

37. **Prove:** The Chain Rule

(If $f(x), g(x)$ are differentiable, then $\frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$)

Proof.

Observe: If $f'(x)$ exists, then $f'(x) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \frac{f(x)-f(x_0)}{x-x_0} + \eta$, where $\eta = \eta(x, x_0)$

and $\eta \to 0$, as $x \to x_0$.

i.e., $f'(x_0) = \frac{f(x)-f(x_0)}{x-x_0} + \eta$, where $\eta \to 0$, as $x \to x_0$.

$\Rightarrow f'(x_0)(x-x_0) = f(x) - f(x_0) + \eta(x-x_0)$, where $\eta \to 0$, as $x \to x_0$.

$\Rightarrow f(x) - f(x_0) = f'(x_0)(x-x_0) - \eta(x-x_0)$, where $\eta \to 0$, as $x \to x_0$.

Letting $g(x)$ play the role of x in the previous equation, we have:

$\Rightarrow f(g(x)) - f(g(x_0)) = f'(g(x_0))(g(x)-g(x_0)) - \eta(g(x)-g(x_0))$, where $\eta \to 0$,

as $g(x) \to g(x_0)$. (eq.1)

Note that since $g(x)$ is continuous (because $g(x)$ is differenitable), $g(x) \to g(x_0)$ as $x \to x_0$.

Hence, eq. 1 can be rendered:

$f(g(x)) - f(g(x_0)) = f'(g(x_0))(g(x)-g(x_0)) - \eta(g(x)-g(x_0))$, where $\eta \to 0$, as $x \to x_0$.

$\Rightarrow \frac{f(g(x))-f(g(x_0))}{x-x_0} = f'(g(x_0))\frac{(g(x)-g(x_0))}{x-x_0} - \eta\frac{(g(x)-g(x_0))}{x-x_0}$, where $\eta \to 0$, as $x \to x_0$.

$\Rightarrow \lim_{x\to x_0} \frac{f(g(x))-f(g(x_0))}{x-x_0} = \lim_{x\to x_0} f'(g(x_0))\frac{(g(x)-g(x_0))}{x-x_0} - \lim_{x\to x_0} \eta\frac{(g(x)-g(x_0))}{x-x_0}$

$$\Rightarrow \underbrace{\lim_{x\to x_0} \frac{f(g(x))-f(g(x_0))}{x-x_0}}_{\frac{d}{dx}[f(g(x))]_{x=x_0}} = \underbrace{\lim_{x\to x_0} f'(g(x_0))}_{f'(g(x_0))} \cdot \underbrace{\lim_{x\to x_0} \frac{(g(x)-g(x_0))}{x-x_0}}_{g'(x_0)} - \underbrace{\lim_{x\to x_0} \eta}_{0} \cdot \underbrace{\lim_{x\to x_0} \frac{(g(x)-g(x_0))}{x-x_0}}_{g'(x_0)}$$

i.e., $\frac{d}{dx}[f(g(x))]_{x=x_0} = f'(g(x_0))\,g'(x_0)$, and in general:

$\frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$ ■

38. **Prove** Rolle's Theorem: Let $f(x)$ be continuous on $[a,b]$ and differentiable on the open interval (a,b), with $f(a) = f(b)$. Then $\exists\ \psi \in (a,b)$ such that $f'(\psi) = 0$.

Proof. By the Extreme Value Theorem, $f(x)$ attains both an absolute maximum value and an absolute minimum value on $[a,b]$.

Case 1: Both the absolute maximum value and the absolute minimum value are attained at the endpoints. Since $f(a) = f(b)$, this implies that

$$\text{Abs min value } = f(a) = f(b) = \text{ Abs max value.}$$

Since

$$\text{Abs min value } \leq f(x) \leq \text{ Abs max value} \qquad \forall x \in [a,b],$$

we have that

$$f(a) \leq f(x) \leq f(a) \qquad \forall x \in [a,b].$$

i.e., $f(x) = f(a) \qquad \forall x \in [a,b]$, which means that $f(x)$ is a constant.
Thus, $f'(x) = 0 \qquad \forall x \in (a,b)$.

Case 2: $f(x)$ attains either an absolute maximum value (or an absolute minimum value) at some point $\psi \in (a,b)$. Note that the point $(\psi, f(\psi))$ is also a relative maximum (or relative minimum). By the preceding problem (problem #36), $f'(\psi) = 0$. ■

39. Prove the Mean Value Theorem: Let $f(x)$ be continuous on $[a,b]$ and differentiable on the open interval (a,b). Then $\exists\ \psi \in (a,b)$ such that $f'(\psi) = \frac{f(b)-f(a)}{b-a}$.

Proof. Consider $F(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]$.

Note that $F(x)$ is continuous, since it is formed from the sums, differences, and products of continuous functions.

Also, note that $F(x)$ is differentiable, since it is formed from the sums, differences, and products of differentiable functions.

Furthermore, $F(a) = 0 = F(b)$.

Thus, Rolle's Theorem guarantees us that $\exists\ \psi \in (a,b)$ such that $F'(\psi) = 0$.

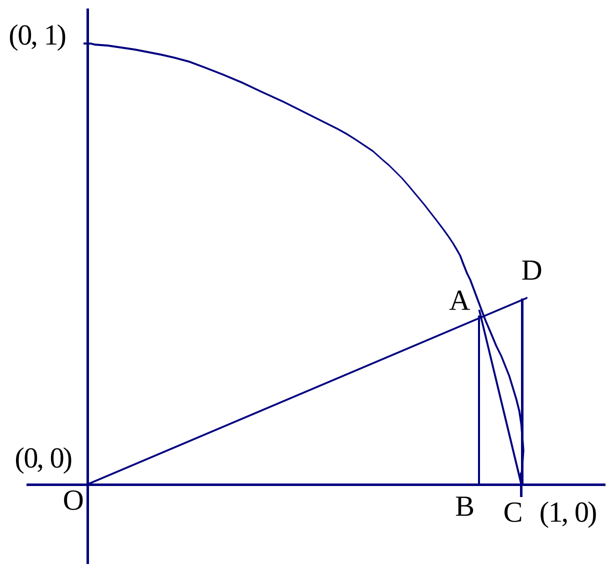
$\Rightarrow 0 = F'(\psi) = f'(\psi) - \frac{f(b)-f(a)}{b-a}(1)$

i.e., $0 = f'(\psi) - \frac{f(b)-f(a)}{b-a}$

$\Rightarrow f'(\psi) = \frac{f(b)-f(a)}{b-a}$ ■

40. **Prove:** $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$; (where x is measured in radians)

Proof.



With reference to the diagram above, we make the following observations:

Referring to ΔOAB :

(a) $\sin(x) = \frac{opp}{hyp} = \frac{\left|\overline{AB}\right|}{1} = \left|\overline{AB}\right|$

i.e., $\left|\overline{AB}\right| = \sin(x)$

(b) $\cos(x) = \frac{adj}{hyp} = \frac{\left|\overline{OB}\right|}{1} = \left|\overline{OB}\right|$

i.e., $\left|\overline{OB}\right| = \cos(x)$

Referring to ΔODC :

(a) $\tan(x) = \frac{opp}{adj} = \frac{\left|\overline{DC}\right|}{1} = \left|\overline{DC}\right|$

i.e., $\left|\overline{DC}\right| = \tan(x)$

Referring to sector OAC :

(a) area sector $OAC = \frac{x}{2}$

(In general, the area of a sector of the unit circle is equal to $\frac{1}{2}$ the length of the arc of the sector.)

Observe: Area $\Delta OAB \leq$ Area sector OAC $\leq$ Area ΔODC

$\Rightarrow \frac{1}{2}\cos(x)\sin(x) \leq \frac{x}{2} \leq \frac{1}{2}\cdot 1 \cdot \tan(x)$

$\Rightarrow \cos(x) \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)}$

$\Rightarrow \cos(x) \leq \frac{\sin(x)}{x} \leq \frac{1}{\cos(x)}$

$\Rightarrow \lim_{\Delta x \to 0} \cos(x) \leq \lim_{\Delta x \to 0} \frac{\sin(x)}{x} \leq \lim_{\Delta x \to 0} \frac{1}{\cos(x)}$

$\Rightarrow 1 \leq \lim_{\Delta x \to 0} \frac{\sin(x)}{x} \leq 1$

$\Rightarrow \lim_{\Delta x \to 0} \frac{\sin(x)}{x} = 1$ ■

41. **Prove:** $\lim_{\Delta x \to 0} \frac{\cos(\Delta x)-1}{\Delta x} = 0$; (where x is measured in radians)

Proof. $\lim_{\Delta x \to 0} \frac{\cos(\Delta x)-1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(\Delta x)-1}{\Delta x} \frac{\cos(\Delta x)+1}{\cos(\Delta x)+1}$

$= \lim_{\Delta x \to 0} \frac{\cos^2(\Delta x)-1}{\Delta x} \frac{1}{\cos(\Delta x)+1} = \lim_{\Delta x \to 0} \frac{\sin^2(\Delta x)}{\Delta x} \frac{1}{\cos(\Delta x)+1}$

$= \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} \frac{\sin(\Delta x)}{\cos(\Delta x)+1} = \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\cos(\Delta x)+1} = 1 \cdot 0 = 0$

i.e., $\lim_{\Delta x \to 0} \frac{\cos(\Delta x)-1}{\Delta x} = 0$ ■

42. **Prove:** $\frac{d}{dx}[\sin(x)] = \cos(x)$ (where x is measured in radians)

Proof. Let $f(x) = \sin(x)$

Observe: $\frac{d}{dx}[\sin(x)] = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x)-\sin(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\sin(x)\cos(\Delta x)+\cos(x)\sin(\Delta x)-\sin(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(x)\cos(\Delta x)-\sin(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{\cos(x)\sin(\Delta x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \sin(x) \frac{\cos(\Delta x)-1}{\Delta x} + \lim_{\Delta x \to 0} \cos(x) \frac{\sin(\Delta x)}{\Delta x}$

$$= \underbrace{\sin(x)}_{\substack{\sin(x)\text{ is a} \\ \text{constant w.r.t } \Delta x}} \cdot \lim_{\Delta x \to 0} \frac{\cos(\Delta x)-1}{\Delta x} + \underbrace{\cos(x)}_{\substack{\cos(x)\text{ is a} \\ \text{constant w.r.t. } \Delta x}} \cdot \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}$$

$= \sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos(x)$

i.e., $\frac{d}{dx}[\sin(x)] = \cos(x)$ ■

43. **Prove:** $\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$ (where x is measured in radians)

Proof. First, we establish a couple of identities:

$\sin\left(x + \frac{\pi}{2}\right) = \sin(x)\cos\left(\frac{\pi}{2}\right) + \cos(x)\sin\left(\frac{\pi}{2}\right) = \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x)$

i.e., $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$

Also:

$\cos\left(x + \frac{\pi}{2}\right) = \cos(x)\cos\left(\frac{\pi}{2}\right) - \sin(x)\sin\left(\frac{\pi}{2}\right) = \cos(x)\cdot 0 - \sin(x)\cdot 1 = -\sin(x)$

i.e., $\cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$

Observe:

$$\frac{d}{dx}\left[\cos(x)\right] = \frac{d}{dx}\left[\sin\left(x + \frac{\pi}{2}\right)\right] = \underbrace{\cos\left(x + \frac{\pi}{2}\right)\cdot 1}_{\text{by chain rule}} = \cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$$

i.e., $\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$ ■