MTH 1126 - Test #3 - 11am Class - Solutions

Spring 2024

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Show CLEARLY how you arrive at your answers.

1. $\int \frac{x-25}{x^2-x-12}dx =$

Note that $\int \frac{x-25}{x^2-x-12}dx$ does not fit the form: $\int \frac{1}{u}du$

Therefore, we decompose $\frac{x-25}{x^2-x-12}$ into the sum of simpler quotients:

1. Make sure that deg(numerator) $\leq$ deg(denominator)

2. Factor the denominator.

$\frac{x-25}{x^2-x-12} = \frac{x-25}{(x+3)(x-4)}$

3. For each linear factor $(x+c)$, form the term $\frac{C_1}{x+c}$

$\frac{x-25}{x^2-x-12} = \frac{x-25}{(x+3)(x-4)} = \frac{C_1}{x+3} + \frac{C_2}{x-4}$

4. Solve for the constants

$\frac{x-25}{(x+3)(x-4)} = \frac{C_1}{x+3} + \frac{C_2}{x-4}$

$\Rightarrow \frac{x-25}{(x+3)(x-4)}(x+3)(x-4) = \frac{C_1}{x+3}(x+3)(x-4) + \frac{C_2}{x-4}(x+3)(x-4)$

i.e., $x-25 = C_1(x-4) + C_2(x+3)$

Plug in "strategic values" of x to find the values of the constants.

$\boxed{x=-3}$

$\Rightarrow -28 = -7C_1$

$\Rightarrow \boxed{C_1 = 4}$

$\boxed{x=4}$

$\Rightarrow -21 = 7C_2$

$\Rightarrow \boxed{C_2 = -3}$

Thus, $\frac{x-25}{(x+3)(x-4)} = \frac{C_1}{x+3} + \frac{C_2}{x-4} = \frac{4}{x+3} - \frac{3}{x-4}$

i.e., $\frac{x-25}{(x+3)(x-4)} = \frac{4}{x+3} - \frac{3}{x-4}$

Consequently, $\int \frac{x-25}{x^2-x-12}dx = \int \left(\frac{4}{x+3} - \frac{3}{x-4}\right) dx = 4\int \frac{1}{x+3}dx - 3\int \frac{1}{x-4}dx$

$$= 4\ln|x+3| - 3\ln|x-4| + C$$

$$\boxed{\int \frac{x-25}{x^2-x-12}dx = 4\ln|x+3| - 3\ln|x-4| + C}$$

2. $\lim_{x\to 0} \frac{1-\cos(x)}{x} =$

   $\lim_{x\to 0} \frac{1-\cos(x)}{x} \sim \frac{0}{0}$ (Use L'Hopital's Rule)

   $\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}[1-\cos(x)]}{\frac{d}{dx}[x]} = \lim_{x\to 0} \frac{\sin(x)}{1} = \frac{0}{1} = 0$

   $$\boxed{\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0}$$

3. $\int \frac{4x^2+2x+48}{x^3+16x} dx =$

Note that $\int \frac{4x^2+2x+48}{x^3+16x} dx$ does not fit the form: $\int \frac{1}{u} du$

Therefore, we decompose $\frac{4x^2+2x+48}{x^3+16x}$ into the sum of simpler quotients:

1. Make sure that deg(numerator) $\leq$ deg(denominator)

2. Factor the denominator.

$\frac{4x^2+2x+48}{x^3+16x} = \frac{4x^2+2x+48}{x(x^2+16)}$ $\quad$ ($x^2 + 16$ is an "irreducible quadratic")

To see this, note that $x^2 \geq 0$. Therefore $x^2 + 16 \geq 16$. And consequently, $x^2 + 16 \neq 0$ for any value of x. This means that $x^2 + 16$ cannot be factored.

Alternatively, note that if we plug the coefficients of $x^2 + 16$ into the quadratic formula, we get: $x = \frac{-0 \pm \sqrt{0^2 - 4(1)(16)}}{2(1)} = \frac{-0 \pm \sqrt{-64}}{2}$.

The fact that we get a negative under the radical tells us that $x^2 + 16$ is irreducible.

3. For each linear factor $(x + c)$, form the term $\frac{C_1}{x+c}$.

3.a. For each irreducible quadratic $ax^2 + bx + c$, form the term $\frac{Ax+B}{ax^2+bx+c}$

$\frac{4x^2+2x+48}{x^3+16x} = \frac{4x^2+2x+48}{x(x^2+16)} = \frac{C}{x} + \frac{Ax+B}{x^2+16}$

4. Solve for the constants

$\frac{4x^2+2x+48}{x(x^2+16)} = \frac{C}{x} + \frac{Ax+B}{x^2+16}$

$\Rightarrow \frac{4x^2+2x+48}{x(x^2+16)} x\left(x^2 + 16\right) = \frac{C}{x} x\left(x^2 + 16\right) + \frac{Ax+B}{x^2+16} x\left(x^2 + 16\right) = C\left(x^2 + 16\right) + (Ax + B)\, x$

i.e., $4x^2 + 2x + 48 = C\left(x^2 + 16\right) + (Ax + B)\, x$

Plug in "strategic values" of x to find the values of the constants.

$\boxed{x = 0}$

$\Rightarrow 48 = 16C$

$\Rightarrow \boxed{C = 3}$

This yields: $4x^2 + 2x + 48 = 3\left(x^2 + 16\right) + (Ax + B)\, x$

Simplifying both sides we have:

$4x^2 + 2x + 48 = (A + 3)\, x^2 + Bx + 48$

Comparing coefficients of x^2 on both sides, we see that:

$4 = A + 3$

$\Rightarrow \boxed{A = 1}$

Comparing coefficients of x on both sides, we see that:

$\boxed{B = 2}$

Thus, $\frac{4x^2+2x+48}{x(x^2+16)} = \frac{3}{x} + \frac{1x+2}{x^2+16} = \frac{3}{x} + \frac{x+2}{x^2+16}$

i.e., $\frac{4x^2+2x+48}{x^3+16x} = \frac{4x^2+2x+48}{x(x^2+16)} = \frac{3}{x} + \frac{x+2}{x^2+16}$

Consequently: $\int \frac{4x^2+2x+48}{x^3+16x}dx = \int \left(\frac{3}{x} + \frac{x+2}{x^2+16}\right) dx = 3\int \frac{1}{x}dx + \int \frac{x+2}{x^2+9}dx$

Remark: $3\int \frac{1}{x}dx = 3\ln|x| + C$, but what about $\int \frac{x+2}{x^2+9}dx$?

$\int \frac{x+2}{x^2+9}dx$ does not fit the form: $\int \frac{1}{u}du$

However, if we split $\int \frac{x+2}{x^2+9}dx$ into two separate integrals:

$\int \frac{x+2}{x^2+9}dx = \int \frac{x}{x^2+9}dx + \int \frac{2}{x^2+9}dx,$

then we get two integrals that we can easily integrate.

Thus, we have:

$$\begin{aligned}
\int \frac{4x^2+2x+48}{x^3+16x}dx &= \int \left(\frac{3}{x} + \frac{x+2}{x^2+16}\right) dx = 3\int \frac{1}{x}dx + \int \frac{x+2}{x^2+9}dx \\
&= 3\int \frac{1}{x}dx + \int \left(\frac{x}{x^2+16} + \frac{2}{x^2+16}\right) dx = 3\int \frac{1}{x}dx + \int \frac{x}{x^2+16}dx + \int \frac{2}{x^2+16}dx \\
&= 3\int \frac{1}{x}dx + \int \underbrace{\frac{1}{x^2+16}}_{\frac{1}{u}} \underbrace{xdx}_{\frac{1}{2}du} + 2\int \underbrace{\frac{1}{x^2+16}}_{\frac{1}{u^2+a^2}} \underbrace{dx}_{du} \\
&= 3\ln|x| + \frac{1}{2}\ln|x^2+16| + 2\left(\frac{1}{4}\arctan\left(\frac{x}{4}\right)\right) + C \\
&= 3\ln|x| + \frac{1}{2}\ln|x^2+16| + \frac{1}{2}\arctan\left(\frac{x}{4}\right) + C
\end{aligned}$$

$$\boxed{\int \frac{4x^2+2x+48}{x^3+16x}dx = 3\ln|x| + \frac{1}{2}\ln|x^2+16| + \frac{1}{2}\arctan\left(\frac{x}{4}\right) + C}$$

4. $\int \sin^3(x)\cos^4(x)\,dx =$

We have an odd power of $\sin(x)$.

1. Reserve a factor of $\sin(x)$ to serve as our "future du."

$$= \int \sin^2(x)\cos^4(x)\underbrace{\sin(x)\,dx}_{\text{"future du"}}$$

This means that we intend to let $u = \cos(x)$

2. Convert remaining sines into cosines

$$= \int \sin^2(x)\cos^4(x)\underbrace{\sin(x)\,dx}_{\text{"future du"}}$$

$$= \int \left(1 - \cos^2(x)\right)\cos^4(x)\sin(x)\,dx$$

$$= \int \left(\cos^4(x) - \cos^6(x)\right)\sin(x)\,dx$$

$$\boxed{\begin{array}{ll} \text{Let} & u = \cos(x) \\ \Rightarrow & \frac{du}{dx} = -\sin(x) \\ \Rightarrow & du = -\sin(x)\,dx \\ & -du = \sin(x)\,dx \end{array}}$$

$$= \int \left(u^4 - u^6\right)(-du)$$

$$= \int \left(u^6 - u^4\right)du$$

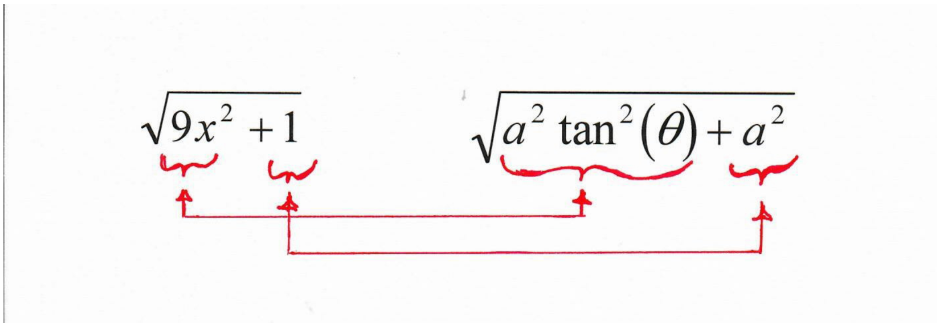
$$= \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C$$

$$= \tfrac{1}{7}\cos^7(x) - \tfrac{1}{5}\cos^5(x) + C$$

$$\boxed{\int \sin^3(x)\cos^4(x)\,dx = \tfrac{1}{7}\cos^7(x) - \tfrac{1}{5}\cos^5(x) + C}$$

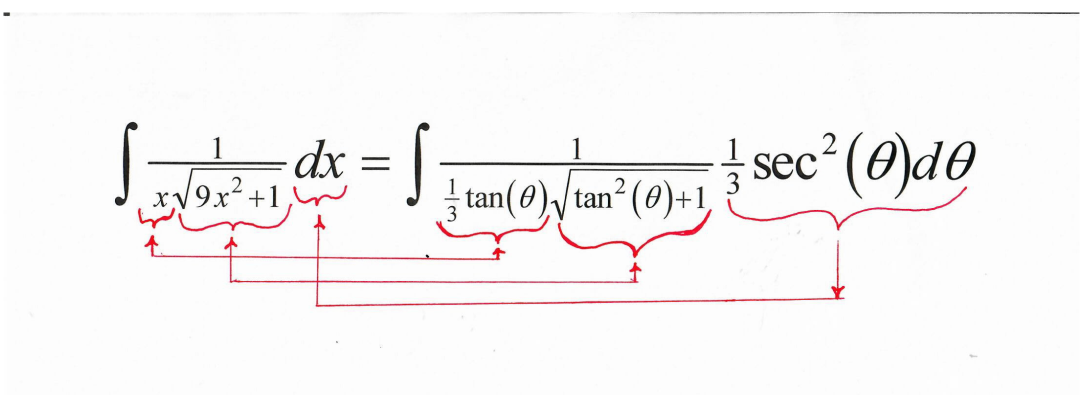
5. $\int \frac{1}{x\sqrt{9x^2+1}}dx =$

We match the radical $\sqrt{9x^2+1}$ with the radical $\sqrt{a^2\tan^2(\theta)+a^2}$



	a^2	$=$	1
$\Rightarrow$	a	$=$	1
	$9x^2$	$=$	$a^2\tan^2(\theta)$
i.e.	$9x^2$	$=$	$\tan^2(\theta)$
$\Rightarrow$	$3x$	$=$	$\tan(\theta)$
$\Rightarrow$	x	$=$	$\frac{1}{3}\tan(\theta)$
$\Rightarrow$	$\frac{dx}{d\theta}$	$=$	$\frac{1}{3}\sec^2(\theta)$
$\Rightarrow$	dx	$=$	$\frac{1}{3}\sec^2(\theta)\,d\theta$

Rewrite the integral in terms of θ

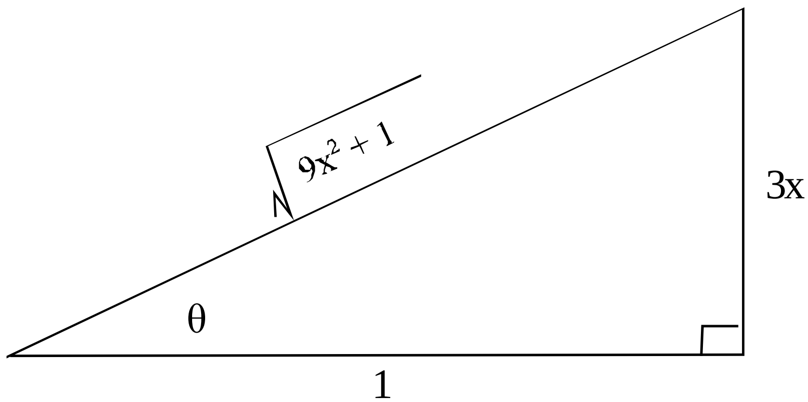


$$\int \frac{1}{x\sqrt{9x^2+1}}dx = \int \frac{1}{\frac{1}{3}\tan(\theta)\sqrt{\tan^2(\theta)+1}}\frac{1}{3}\sec^2(\theta)\,d\theta = \int \frac{1}{\frac{1}{3}\tan(\theta)\sqrt{\sec^2(\theta)}}\frac{1}{3}\sec^2(\theta)\,d\theta$$

$$= \int \frac{1}{\frac{1}{3}\tan(\theta)\sec(\theta)}\frac{1}{3}\sec^2(\theta)\,d\theta = \int \frac{1}{\tan(\theta)}\sec(\theta)\,d\theta = \int \cot(\theta)\sec(\theta)\,d\theta$$

$$= \int \frac{\cos(\theta)}{\sin(\theta)}\frac{1}{\cos(\theta)}d\theta = \int \frac{1}{\sin(\theta)}d\theta = \int \csc(\theta)\,d\theta = \ln|\csc(\theta)-\cot(\theta)|+C$$

To convert back to x, recall that $x = \frac{1}{3}\tan(\theta)$ (i.e., $\tan(\theta) = \frac{3x}{1} = \frac{\text{opp}}{\text{adj}}$)



$$\int \frac{1}{x\sqrt{9x^2+1}}dx = \ldots = \ln|\csc(\theta) - \cot(\theta)| + C = \ln\left|\frac{\sqrt{9x^2+1}}{3x} - \frac{1}{3x}\right| + C$$

$$\boxed{\int \frac{1}{x\sqrt{9x^2+1}}dx = \ln\left|\frac{\sqrt{9x^2+1}}{3x} - \frac{1}{3x}\right| + C}$$

6. $\lim_{x\to+\infty} \frac{\ln(x)}{x^2} =$

$\lim_{x\to+\infty} \frac{\ln(x)}{x^2} \sim \frac{-\infty}{\infty}$ We can use L'Hôpital's Rule

$\lim_{x\to+\infty} \frac{\ln(x)}{x^2} = \lim_{x\to+\infty} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[x^2]} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to+\infty} \frac{1}{x^2} = 0$

i.e., $\lim_{x\to+\infty} \frac{\ln(x)}{x^2} = 0$

7. $\int x\cos(2x)\,dx =$

Use Integration by Parts

Our "Rules of Thumb" apply

Let u be the portion of the integrand whose derivative is simpler that itself.

$\Rightarrow u = x$

Let dv be the most complicated portion of the integrand that can be integrated

$\Rightarrow dv = \cos(2x)$

$$\begin{array}{llll} & u = x & & dv = \cos(2x)\,dx \\ \Rightarrow & \frac{du}{dx} = 1 & \Rightarrow & v = \int \cos(2x)\,dx \\ \Rightarrow & du = dx & \Rightarrow & v = \frac{1}{2}\sin(2x) \end{array}$$

$$\begin{aligned} \Rightarrow \int x\cos(2x)\,dx = \int udv = uv - \int vdu &= x\left(\tfrac{1}{2}\sin(2x)\right) - \int \left(\tfrac{1}{2}\sin(2x)\right)dx \\ &= \tfrac{1}{2}x\sin(2x) - \tfrac{1}{2}\int \sin(2x) = \tfrac{1}{2}x\sin(2x) - \tfrac{1}{2}\left[-\tfrac{1}{2}\cos(2x)\right] + C \\ &= \tfrac{1}{2}x\sin(2x) + \tfrac{1}{4}\cos(2x) + C \end{aligned}$$

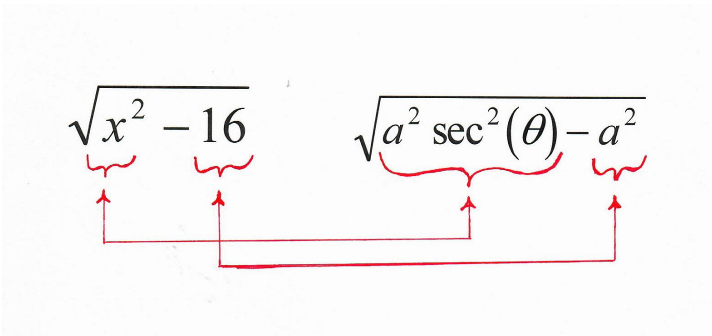
$\int x\cos(2x) = \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$

WOW! Extra (10 pts - all or nothing)

$\int \frac{\sqrt{x^2-16}}{x} dx =$

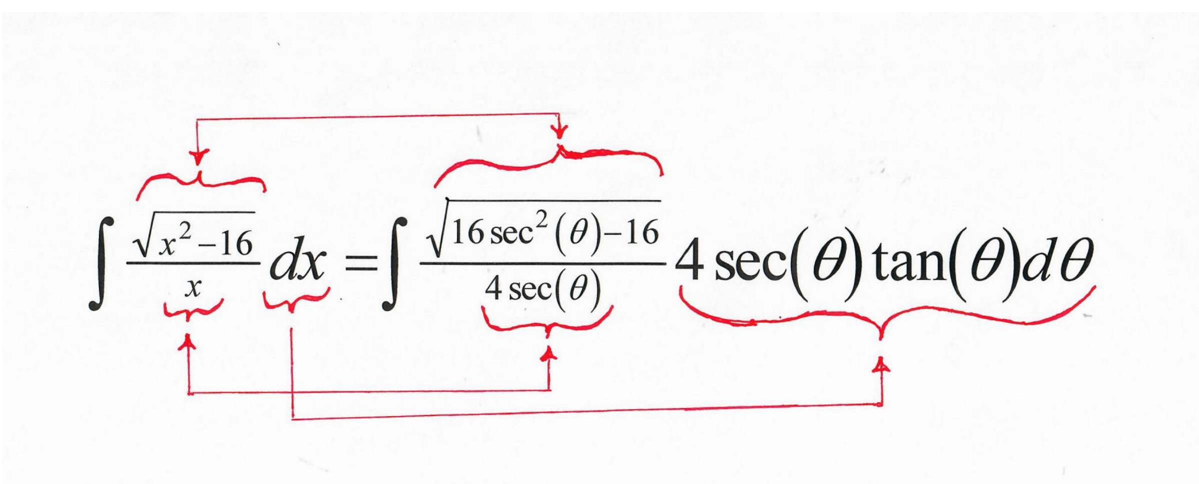
Compute: $\int \frac{\sqrt{x^2-16}}{x} dx =$

We match the radical $\sqrt{x^2 - 16}$ with the radical $\sqrt{a^2 \sec^2(\theta) - a^2}$



	a^2	$=$	16
$\Rightarrow$	a	$=$	4
	x^2	$=$	$a^2 \sec^2(\theta)$
i.e.	x^2	$=$	$16 \sec^2(\theta)$
$\Rightarrow$	x	$=$	$4 \sec(\theta)$
$\Rightarrow$	$\frac{d}{dx}[x]$	$=$	$\frac{d}{dx}[4 \sec(\theta)]$
$\Rightarrow$	1	$=$	$4 \sec(\theta) \tan(\theta) \frac{d\theta}{dx}$
$\Rightarrow$	dx	$=$	$4 \sec(\theta) \tan(\theta)\, d\theta$

Rewrite the integral in terms of θ



$\int \frac{\sqrt{x^2-16}}{x} dx = \int \frac{\sqrt{16\sec^2(\theta)-16}}{4\sec(\theta)} 4 \sec(\theta) \tan(\theta)\, d\theta = \int \frac{\sqrt{16\tan^2(\theta)}}{1} \tan(\theta)\, d\theta$

$= \int 4 \tan(\theta) \cdot \tan(\theta)\, d\theta = 4 \int \tan^2(\theta)\, d\theta = 4 \int (\sec^2(\theta) - 1)\, d\theta$

$= 4\left(\tan(\theta) - \theta\right) + C$

Now we must re-express the answer in terms of the original variable x.

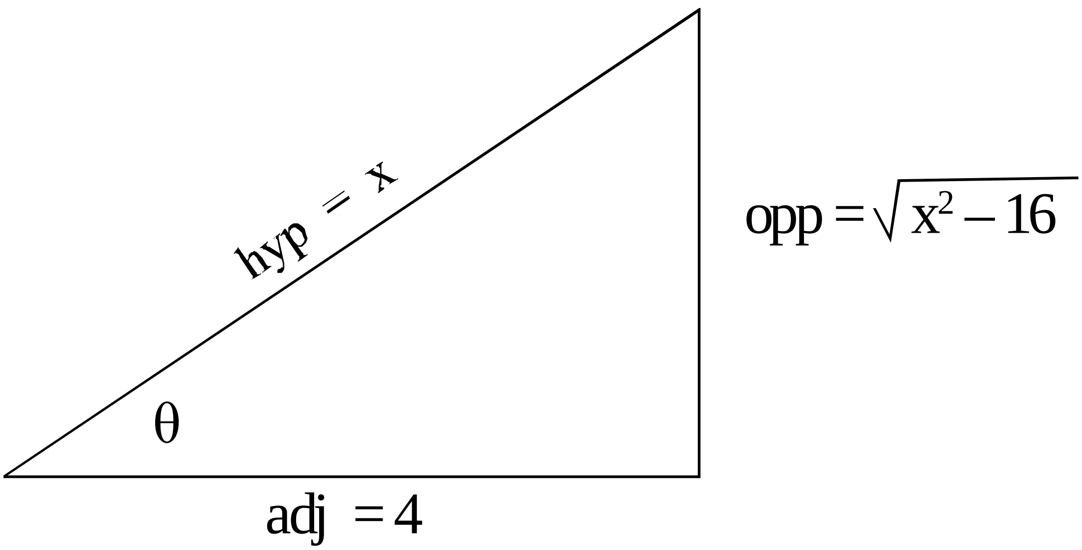
(Continued)

Recall: $x = 4\sec(\theta)$

$\Rightarrow \frac{x}{4} = \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\left(\frac{\text{adj}}{\text{hyp}}\right)} = \frac{\text{hyp}}{\text{adj}}$

i.e., $\frac{x}{4} = \frac{\text{hyp}}{\text{adj}}$

We will draw a right triangle that depicts this relationship.



From the diagram above, $\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x^2-16}}{4}$

i.e., $\tan(\theta) = \frac{\sqrt{x^2-16}}{4}$

To re-express θ in terms of x, we consider the equation $x = 4\sec(\theta)$

$\Rightarrow \frac{x}{4} = \sec(\theta)$

$\Rightarrow \sec^{-1}\left(\frac{x}{4}\right) = \sec^{-1}(\sec(\theta))$

$\Rightarrow \sec^{-1}\left(\frac{x}{4}\right) = \theta$

i.e., $\theta = \sec^{-1}\left(\frac{x}{3}\right)$

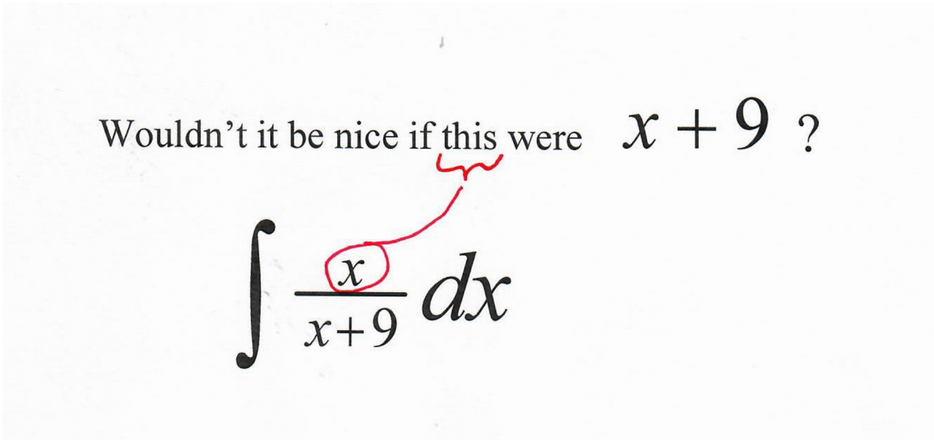
Thus, $\int \frac{\sqrt{x^2-16}}{x} dx = \ldots = 4\left(\tan(\theta) - \theta\right) + C = 4\left(\frac{\sqrt{x^2-16}}{4} - \sec^{-1}\left(\frac{x}{4}\right)\right) + C$

i.e., $\int \frac{\sqrt{x^2-16}}{x} dx = 4\left(\frac{\sqrt{x^2-16}}{4} - \sec^{-1}\left(\frac{x}{4}\right)\right) + C = \sqrt{x^2-16} - 4\sec^{-1}\left(\frac{x}{4}\right) + C$

$$\int \frac{\sqrt{x^2-16}}{x} dx = 4\left(\frac{\sqrt{x^2-16}}{4} - \sec^{-1}\left(\frac{x}{4}\right)\right) + C = \sqrt{x^2-16} - 4\sec^{-1}\left(\frac{x}{4}\right) + C$$

WOW! Extra (5 pts - all or nothing)

$\int \frac{x}{x+9} dx =$



$\int \frac{x}{x+9} dx = \int \left(\frac{x+9}{x+9} - \frac{9}{x+9} \right) dx = \int \left(1 - \frac{9}{x+9}\right) dx = \int 1 dx - \int \frac{9}{x+9} dx$

$= \int 1 dx - 9 \int \frac{1}{x+9} dx = x - 9 \ln |x+9| + C$

$$\boxed{\int \frac{x}{x+9} dx = x - 9 \ln |x+9| + C}$$