MTH 1125 (9 am) Test #3 – Solutions

Fall 2016

Pat Rossi Name ______________________________

Show CLEARLY how you arrive at your answers.

1. $f(x) = 2x^3 + 3x^2 - 36x$. Identify the intervals on which $f(x)$ is increasing/decreasing, and identify all relative maximums and minimums.

   i. Compute $f'(x)$ and find critical numbers

   $f'(x) = 6x^2 + 6x - 36$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 6x^2 + 6x - 36 = 0$

   $\Rightarrow 6x^2 + 6x - 36 = 0$

   $\Rightarrow x^2 + x - 6 = 0$

   $\Rightarrow (x + 3)(x - 2) = 0$

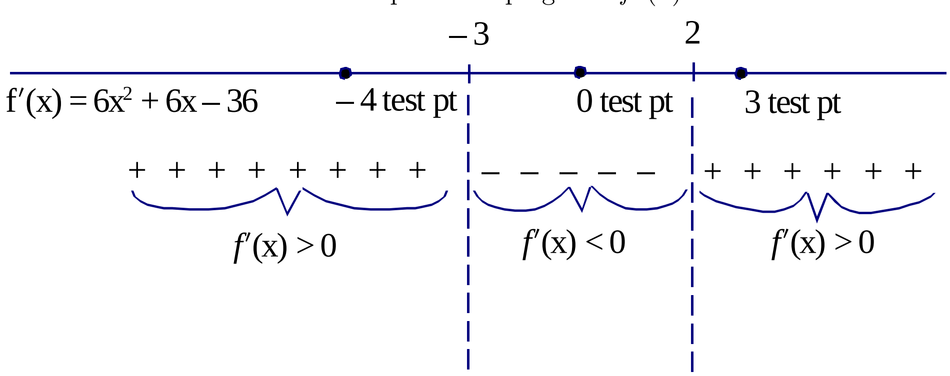
   $\Rightarrow x = -3$; $x = 2$ critical numbers

   b. "Type b" $(f'(c)$ undefined$)$

   There are none.

   ii. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

   iii. From each interval select a "test point" to plug into $f'(x)$



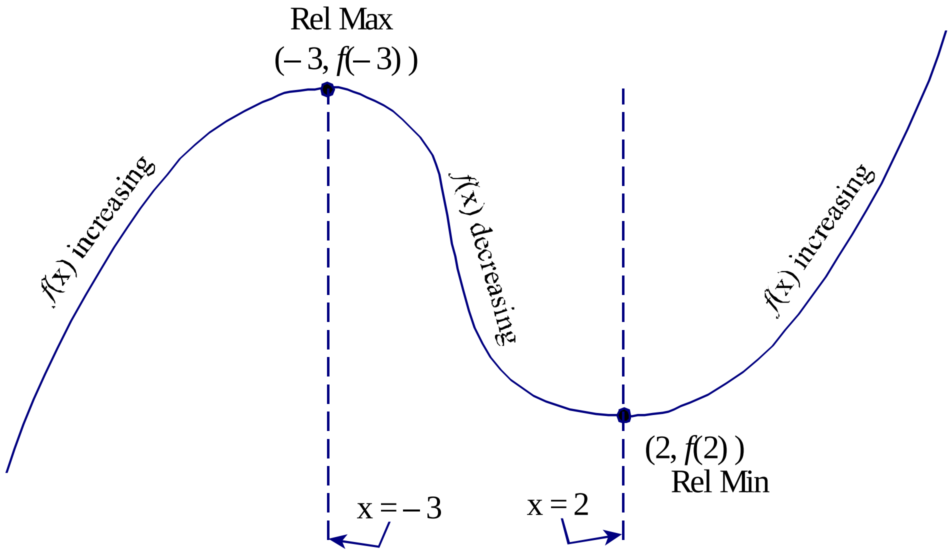
$f(x)$ is **increasing** on the intervals $(-\infty, -3)$ and $(2, \infty)$

(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval $(-3, 2)$

(Because $f'(x)$ is negative on this interval)

iv. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



$f(x)$ is **increasing** on the intervals $(-\infty, -3)$ and $(2, \infty)$

(Because $f'(x)$ is **negative)**

$f(x)$ is **decreasing** on the interval $(-3, 2)$

(Because $f'(x)$ is **positive)**

$(-3, f(-3)) = (-3, 81)$ Relative Max

$(2, f(2)) = (2, -44)$ Relative Min

2. $f(x) = 4x^{\frac{9}{5}} + 9x^{\frac{4}{5}} + \frac{1}{2}$. Identify the intervals on which $f(x)$ is increasing/decreasing, and identify all relative maximums and minimums.

i. Compute $f'(x)$ and find critical numbers

$f'(x) = \frac{36}{5}x^{\frac{4}{5}} + \frac{36}{5}x^{-\frac{1}{5}} = \frac{36x^{\frac{4}{5}}}{5} + \frac{36}{5x^{\frac{1}{5}}} = \frac{36x^{\frac{4}{5}}}{5}\frac{x^{\frac{1}{5}}}{x^{\frac{1}{5}}} + \frac{36}{5x^{\frac{1}{5}}} = \frac{36x}{5x^{\frac{1}{5}}} + \frac{36}{5x^{\frac{1}{5}}} = \frac{36x+36}{5x^{\frac{1}{5}}}$

i.e., $f'(x) = \frac{36x+36}{5x^{\frac{1}{5}}}$

a. "Type a" $(f'(c) = 0)$

Set $f'(x) = \frac{36x+36}{5x^{\frac{1}{5}}} = 0$

$\Rightarrow 36x + 36 = 0$

$\Rightarrow x + 1 = 0$

$\Rightarrow x = -1$ critical numbers

b. "Type b" $(f'(c)$ undefined$)$

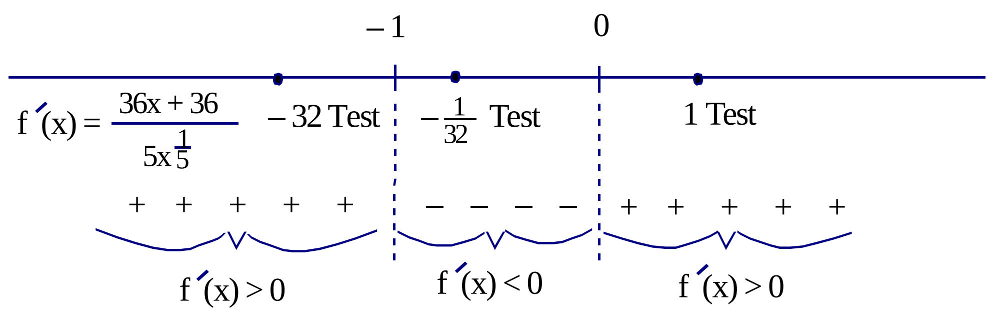
Set denominator $5x^{\frac{1}{5}} = 0$

$\Rightarrow x = 0$ critical number

ii. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

iii. From each interval select a "test point" to plug into $f'(x)$

b ex 3a



3.pdf

$f(x)$ is **increasing** on the intervals $(-\infty, -1)$ and $(0, \infty)$

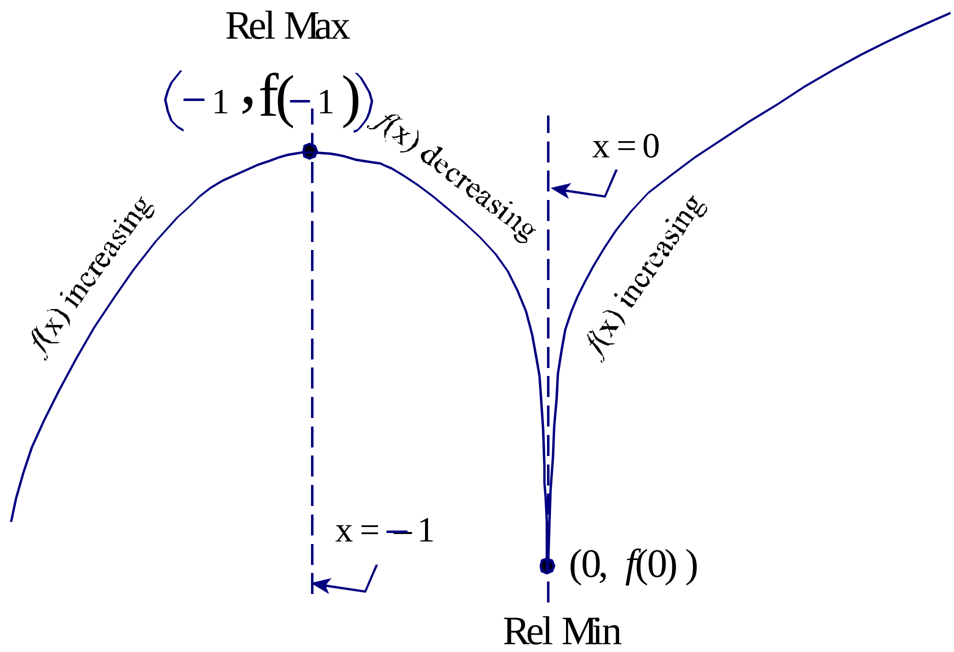
(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the intervals $(-1, 0)$

(Because $f'(x)$ is negative on this interval.

iv. To find the relative maxes and mins, sketch a rough graph of $f(x)$.

b ex 3b



$_{4}.pdf$

$f(x)$ is **increasing** on the intervals $(-\infty, -1)$ and $(0, \infty)$

$f(x)$ is **decreasing** on the interval $(-1, 0)$

Relative Max $\quad (-1, f(-1)) = \left(-1, \frac{11}{2}\right)$

Relative Min $\quad (0, f(0)) = \left(0, \frac{1}{2}\right)$

3. $f(x) = x^3 + 3x^2 - 24x$ on the interval $[-2, 3]$. Find the absolute maximum value and absolute minimum value of $f(x)$.

Since $f(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[-2, 3]$, we can use the Absolute Max/Min Value Test

i. Compute $f'(x)$ and find the critical numbers

$f'(x) = 3x^2 + 6x - 24$

a. "Type a" ($f'(c) = 0$)

Set $f'(x) = 3x^2 + 6x - 24 = 0$

$\Rightarrow 3x^2 + 6x - 24 = 0$

$\Rightarrow x^2 + 2x - 8 = 0$

$\Rightarrow (x+4)(x-2) = 0$

$\Rightarrow x = -4$; $x = 2$ "type a" crit. numbers

Since $x = -4$ is not in the interval $[-2, 3]$, we discard $x = -4$ as a critical number.

b. "Type b" ($f'(c)$ is undefined)

No "type b" critical numbers.

ii. Plug critical numbers and endpoints into the original function.

$f(-2) = (-2)^3 + 3(-2)^2 - 24(-2) = 52 \leftarrow$ Abs Max Value

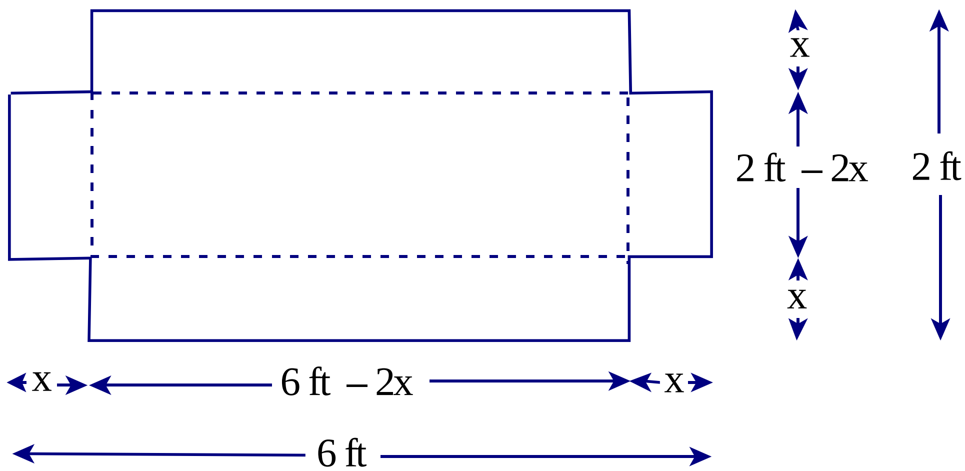
$f(2) = (2)^3 + 3(2)^2 - 24(2) = -28 \leftarrow$ Abs MinValue

$f(3) = (3)^3 + 3(3)^2 - 24(3) = -18$

Abs Max Value $= 52$ (attained at $x = -2$) Abs Min Value $= -28$ (attained at $x = 2$)

From Exercises 4 and 5, select one.

4. An open box is to be constructed from a rectangular piece of cardboard of length 6 ft and width 2 ft, by cutting equal sized squares from the corners (as shown below), and bending up the sides. What are the dimensions of the box of largest volume that can be constructed in this manner?



1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Volume** of the box, $V = l \cdot w \cdot h = (6\text{ft} \ - 2x)\,(2\text{ft} \ - 2x)\,x = 12\text{ft}^2 \ x - 16\text{ft} \ x^2 + 4x^3$

   i.e., $V\,(x) = 4x^3 - 16\text{ft} \ x^2 + 12\text{ft}^2 \ x$

   a. Draw a picture where relevant.

   1. 1. (Done)

2. Express V as a function of one other variable.

   (Done)

3. Determine the restrictions on the independent variable x.

   From the picture, $0 \text{ ft} \leq x \leq 1 \text{ ft}$

4. Maximize $V(x)$, using the techniques of Calculus.

   Note that $V(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0 \text{ ft}, 1 \text{ ft}]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $V'(x) = 12x^2 - 32\text{ft } x + 12\text{ft}^2$

      a. "Type a" $(V'(c) = 0)$

         $\Rightarrow V'(x) = 12x^2 - 32\text{ft } x + 12\text{ft}^2 = 0$

         $\Rightarrow 12x^2 - 32\text{ft } x + 12\text{ft}^2 = 0$

         $\Rightarrow 3x^2 - 8\text{ft } x + 3\text{ft}^2 = 0$ This does not factor (sorry!)

         $x = \frac{-(-8)\pm\sqrt{(-8)^2-(4)(3)(3)}}{2(3)} = \frac{8\pm\sqrt{64-36}}{6} = \frac{8\pm\sqrt{28}}{6} = \frac{8\pm\sqrt{(4)(7)}}{6} = \frac{8\pm\sqrt{4}\sqrt{7}}{6} = \frac{8\pm 2\sqrt{7}}{6}$

         $= \frac{4\pm\sqrt{7}}{3}$

         $\Rightarrow x = \frac{4-\sqrt{7}}{3}\text{ft};\ x = \frac{4+\sqrt{7}}{3}\text{ft}$ (crit. numbers)

         We discard $x = \frac{4+\sqrt{7}}{3}\text{ft}$, since it is not in the interval $[0\text{ft}, 1\text{ft}]$

      b. "Type b" ($V'(c)$ is undefined)

         Look for x-values that cause division by zero in $V'(x)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $V(0\text{ft}) = 4(0\text{ft})^3 - 16\text{ft }(0\text{ft})^2 + 12\text{ft}^2 (0\text{ft}) = 0\text{ft}^2$

      $V\left(\frac{4-\sqrt{7}}{3}\text{ft}\right) = 4\left(\frac{4-\sqrt{7}}{3}\text{ft}\right)^3 - 16\text{ft}\left(\frac{4-\sqrt{7}}{3}\text{ft}\right)^2 + 12\text{ft}^2\left(\frac{4-\sqrt{7}}{3}\text{ft}\right) = \left(\frac{56\sqrt{7}-80}{27}\right)\text{ft}^3 \longleftarrow$ Abs Max Value

      $V(1\text{ft}) = 4(1\text{ft})^3 - 16\text{ft }(1\text{ft})^2 + 12\text{ft}^2 (1\text{ft}) = 0\text{ft}^2$

5. Make sure that we've answered the original question.

   "What are the dimensions of the box of largest volume?"

   $x = \frac{4-\sqrt{7}}{3}\text{ft}$

   Length $= 6\text{ft} - 2x = 6\text{ft} - 2\left(\frac{4-\sqrt{7}}{3}\text{ft}\right) = \frac{10+2\sqrt{7}}{3}\text{ft}$

   Width $= 2\text{ft} - 2x = 2\text{ft} - 2\left(\frac{4-\sqrt{7}}{3}\text{ft}\right) = \frac{-2+2\sqrt{7}}{3}\text{ft}$

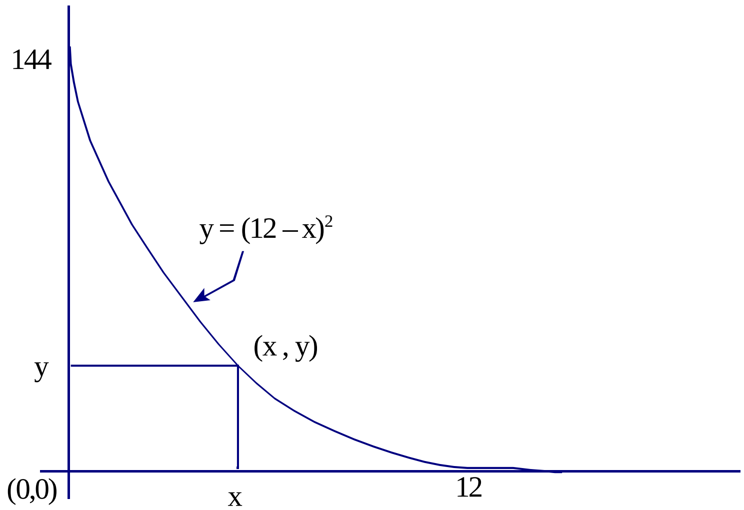
   Height $= x = \frac{4-\sqrt{7}}{3}\text{ft}$

   Length $= \frac{10+2\sqrt{7}}{3}\text{ft}$

   Width $= \frac{-2+2\sqrt{7}}{3}\text{ft}$

   Height $= \frac{4-\sqrt{7}}{3}\text{ft}$

5. A rectangle is inscribed within the region bounded by the positive x and y axes, and the graph of $y = (12 - x)^2$ (as shown below).What should the value of x be in order to make the area of the rectangle as large as possible?



1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = xy$

   a. Draw a picture where relevant.

   1. 1. (Done)

2. Express A as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that the point (x, y) must be on the graph of $f(x) = (12 - x)^2$.

   Hence, the y-coordinate of the point (x, y) is $y = (12 - x)^2$.

   Plug this into the equation $A = xy$

   $\Rightarrow A(x) = x(12 - x)^2 = x^3 - 24x^2 + 144x$

   i.e., $A(x) = x^3 - 24x^2 + 144x$

3. Determine the restrictions on the independent variable x.

   From the picture, $0 \leq x \leq 12$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0,12]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = 3x^2 - 48x + 144$

      a. "Type a" $(f'(c) = 0)$

         $\Rightarrow A'(x) = 3x^2 - 48x + 144 = 0$

         $\Rightarrow 3x^2 - 48x + 144 = 0$

         $\Rightarrow x^2 - 16x + 48 = 0$

         $\Rightarrow (x-4)(x-12) = 0$

         $\Rightarrow x = 4$ and $x = 12$ are critical numbers

      b. "Type b" ($f'(c)$ is undefined)

         Look for x-values that cause division by zero in $f'(x)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0) = (0)^3 - 24(0)^2 + 144(0) = 0$

      $A(4) = (4)^3 - 24(4)^2 + 144(4) = 256 \longleftarrow$ Abs Max Value

      $A(12) = (12)^3 - 24(12)^2 + 144(12) = 0$

5. Make sure that we've answered the original question.

   "Determine the value of x ..."

   $\boxed{x = 4}$