MTH 1126 - Test #2 (makeup) - Solutions

SPRING, 2005

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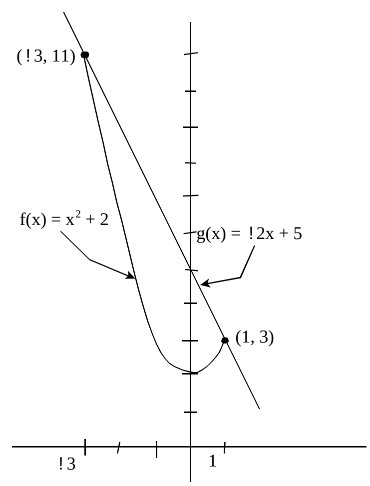
Instructions. Show CLEARLY how you arrive at your answers.

1. Use the "$f - g$" method to compute the area bounded by the graphs of the functions $f(x) = x^2 + 2$; and $g(x) = -2x + 5$.

    1. First, graph the functions.



Note that the bounded region spans the interval $[-3, 1]$ on the x-axis, and that over the bounded region, $g(x) = -2x + 5 \geq x^2 + 2$.

Hence, the area of the bounded region is given by:

$\int_{-3}^{1} [(-2x + 5) - (x^2 + 2)]\, dx = \int_{-3}^{1} (3 - x^2 - 2x)\, dx = \left[3x - \frac{1}{3}x^3 - x^2\right]_{-3}^{1} =$

$\left(3(1) - \frac{1}{3}(1)^3 - (1)^2\right) - \left(3(-3) - \frac{1}{3}(-3)^3 - (-3)^2\right) = \frac{32}{3}$

i.e., the area of the bounded region is $\frac{32}{3}$.

2. Compute the arclength of the graph of the function $f(x) = x^{\frac{3}{2}} + 6$ from the point $(1, 7)$ and $(4, 14)$.

Use the arclength formula: Arclength $= \int_{x=a}^{x=b} \sqrt{1 + (f'(x))^2}\, dx$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}}$

$(f'(x))^2 = \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2 = \frac{9}{4}x$

$$\text{Arclength} = \int_{x=a}^{x=b} \sqrt{1+\left(f'(x)\right)^2}\,dx = \int_{x=1}^{x=4} \sqrt{1+\tfrac{9}{4}x}\,dx = \int_{x=1}^{x=4} \underbrace{\left(1+\frac{9}{4}x\right)^{\frac{1}{2}}}_{u^{\frac{1}{2}}} \underbrace{dx}_{\frac{4}{9}du} =$$

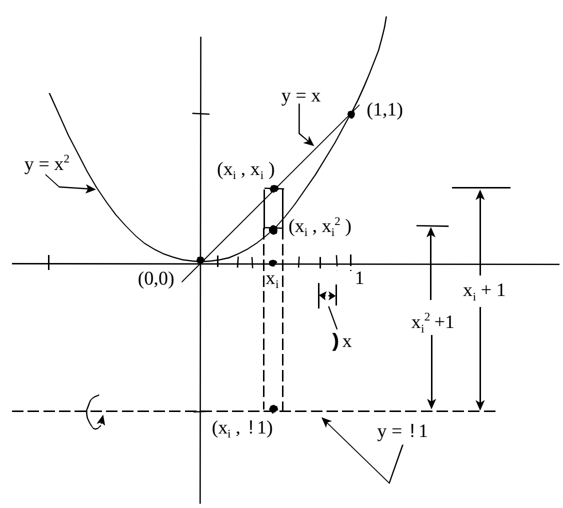
$$\int_{u=\frac{13}{4}}^{u=10} u^{\frac{1}{2}}\tfrac{4}{9}du = \tfrac{4}{9}\int_{u=\frac{13}{4}}^{u=10} u^{\frac{1}{2}}du = \tfrac{4}{9}\left[\tfrac{2}{3}u^{\frac{3}{2}}\right]_{u=\frac{13}{4}}^{u=10} = \tfrac{8}{27}\left[(10)^{\frac{3}{2}} - \left(\tfrac{13}{4}\right)^{\frac{3}{2}}\right]$$

$$\boxed{\begin{array}{llll} \text{Let} & u & = & 1+\frac{9}{4}x \\ \Rightarrow & \frac{du}{dx} & = & \frac{9}{4} \\ \Rightarrow & du & = & \frac{9}{4}dx \\ \Rightarrow & \frac{4}{9}du & = & dx \\ & & & \text{When } x=1;\ u = 1+\frac{9}{4}(1) = \frac{13}{4} \\ & & & \text{When } x=4;\ u = 1+\frac{9}{4}(4) = 10 \end{array}}$$

For problems 3 to 5, use the "five step method" (partition the interval, form the sum, take the limit)

3. Use the "Disc method" to compute the volume of the solid of revolution generated by revolving the region bounded by the graph $f(x) = x^2$, $g(x) = x$ and the x-axis about the line $y = -1$.

    1. Graph the bounded region



    2. Partition the interval spanned by the bounded region, and inscribe rectangles perpendicular to the axis of revolution.

3. Revolve the i^{th} rectangle about the axis of revolution.

$$\begin{aligned}
Vol.\ i^{th}\text{ large disc} &= \pi R_i^2 \Delta x \\
&= \pi \left(x_i+1\right)^2 \Delta x \\
&= \pi \left(x_i^2+2x_i+1\right) \Delta x \\
\text{Vol. } i^{th} \text{ hole} &= \pi r_i^2 \Delta x \\
&= \pi \left(x_i^2+1\right)^2 \Delta x \\
&= \pi \left(x_i^4+2x_i^2+1\right)^2 \Delta x \\
\text{Vol. } i^{th} \text{ washer} &= \left(Vol.\ i^{th}\ large\ disc\right) - \left(Vol.\ i^{th}\, hole\right) \\
&= \pi \left(x_i^2+2x_i+1\right) \Delta x - \pi \left(x_i^4+2x_i^2+1\right)^2 \Delta x \\
&= \pi \left(2x_i - x_i^2 - x_i^4\right) \Delta x
\end{aligned}$$

4. Approximate the volume of the solid of revolution by adding up the volumes of the washers.

   $Vol \approx \sum_{i=1}^{n} \pi \left(2x_i - x_i^2 - x_i^4\right) \Delta x$

5. Let $\Delta x \to \infty$

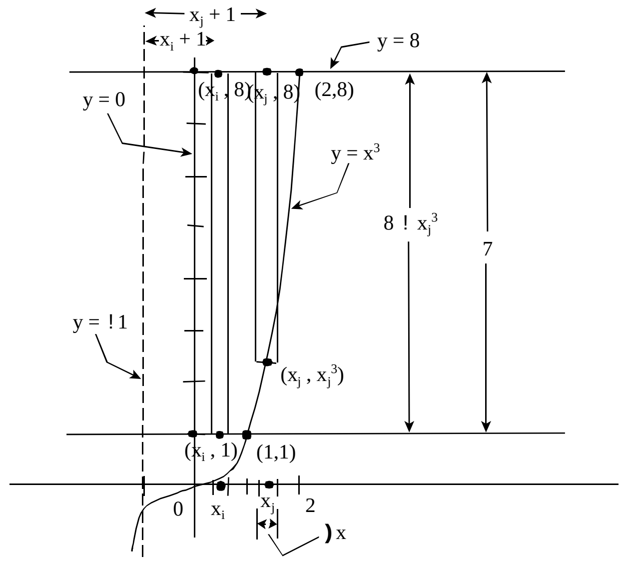
   $Vol = \lim_{\Delta x \to \infty} \sum_{i=1}^{n} \pi \left(2x_i - x_i^2 - x_i^4\right) \Delta x = \int_0^1 \pi \left(2x - x^2 - x^4\right) dx$

   $= \pi \left[x^2 - \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi \left[\left((1)^2 - \frac{1}{3}(1)^3 - \frac{1}{5}(1)^5\right) - \left((0)^2 - \frac{1}{3}(0)^3 - \frac{1}{5}(0)^5\right)\right]$

   $= \frac{7\pi}{15}.$

4. Use the “shell method” to compute the volume of the solid of revolution generated by revolving the region bounded by the graph $f(x) = x^3$, the y-axis, and the line $y = 8$, and $y = 1$ about the line line $x = -1$. (consider only the region that lies to the right of the y-axis).

   1. Graph the bounded region



2. Revolve the i^{th} rectangle about the axis of revolution to form the i^{th} shell.

   (Note that the definition of the i^{th} rectangle changes at $x = 1$.)

$$\begin{aligned}
\text{Vol. } i^{th} \text{ shell} &= 2\pi R_i h_i \Delta x \text{ for } 0 \leq x \leq 1 \\
&= 2\pi (x_i + 1)(7) \Delta x \text{ for } 0 \leq x \leq 1 \\
&= 14\pi (x_i + 1) \Delta x \text{ for } 0 \leq x \leq 1 \\
\text{Vol. } j^{th} \text{ shell} &= 2\pi R_j h_j \Delta x \text{ for } 1 \leq x \leq 2 \\
&= 2\pi (x_j + 1)\left(8 - x_j^3\right) \Delta x \text{ for } 1 \leq x \leq 2 \\
&= 2\pi \left(8x_j + 8 - x_j^4 - x_j^3\right) \Delta x \text{ for } 1 \leq x \leq 2
\end{aligned}$$

3. Approximate the volume of the solid of revolution by adding the volumes of the shells.

   Vol. of Solid $\approx$ (Vol. of Shells from $x = 0$ to $x = 1$)+(Vol. of Shells from $x = 1$ to $x = 2$)

   $\approx \sum_{i=1}^{n} 14\pi (x_i + 1) \Delta x + \sum_{i=1}^{n} 2\pi \left(8x_j + 8 - x_j^4 - x_j^3\right) \Delta x$

4. Let $\Delta x \to 0$

   Vol $= \lim_{\Delta x \to 0} \sum_{i=1}^{n} 14\pi (x_i + 1) \Delta x + \lim_{\Delta x \to 0} \sum_{i=1}^{n} 2\pi \left(8x_j + 8 - x_j^4 - x_j^3\right) \Delta x$

   $= \int_0^1 14\pi (x + 1)\, dx + \int_1^2 2\pi \left(8x + 8 - x^4 - x^3\right) \Delta x$

   $= 14\pi \left[\frac{1}{2}x^2 + 1\right]_0^1 + 2\pi \left[4x^2 + 8x - \frac{1}{5}x^5 - \frac{1}{4}x^4\right]_1^2$

   $= 14\pi \left[\left(\frac{1}{2}(1)^2 + 1\right) - \left(\frac{1}{2}(0)^2 + 1\right)\right]$

   $+2\pi \left[\left(4(2)^2 + 8(2) - \frac{1}{5}(2)^5 - \frac{1}{4}(2)^4\right) - \left(4(1)^2 + 8(1) - \frac{1}{5}(1)^5 - \frac{1}{4}(1)^4\right)\right]$

$= 7\pi + \frac{201\pi}{10} = \frac{271\pi}{10}$

5. $\int_0^5 (f(x) - g(x))\, dx = 8$ and $\int_0^5 2f(x)\, dx = 8$. Compute $\int_0^5 g(x)\, dx$.

$$\begin{aligned}
\text{Observe:} \quad & 8 = \int_0^5 2f(x)\, dx = 2\int_0^5 f(x)\, dx \\
\text{i.e.,} \quad & 2\int_0^5 f(x)\, dx = 8 \\
\Rightarrow \quad & \int_0^5 f(x)\, dx = 4 \\
\text{Next, observe that} \quad & \int_0^5 (f(x) - g(x))\, dx = \int_0^5 f(x)\, dx - \int_0^5 g(x)\, dx \\
\Rightarrow \quad & \int_0^5 g(x)\, dx = \int_0^5 f(x)\, dx - \int_0^5 (f(x) - g(x))\, dx = 4 - 8 = -4 \\
\text{i.e.,} \quad & \int_0^5 g(x)\, dx = -4
\end{aligned}$$