Exercises Involving Real Numbers #1 - Solutions

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Instructions. Prove:

1. For all real numbers a, b, c; $a < b \Leftrightarrow a + c < b + c$

   Proof.

   $\boxed{a < b \Rightarrow a + c < b + c}$

   Let the hypothesis be given. (i.e., suppose that $a < b$)

   $\Rightarrow b - a$ is positive

   $\Rightarrow b - a + \underbrace{(c - c)}_{=0}$ is positive

   $\Rightarrow (b + c) - (a + c)$ is positive

   $\Rightarrow a + c < b + c$

   $\boxed{a + c < b + c \Rightarrow a < b}$

   Let the hypothesis be given. (i.e., suppose that $a + c < b + c$)

   $\Rightarrow (b + c) - (a + c)$ is positive.

   $\Rightarrow b + c - a - c$ is positive

   $\Rightarrow b - a + (c - c)$ is positive

   $\Rightarrow b - a$ is positive ■

2. For all real numbers a, b, c; $a < b \Leftrightarrow a - c < b - c$

   $\boxed{a < b \Rightarrow a - c < b - c}$

   Let the hypotheses be given.

   Proof. From part 1, we have:

   For all real numbers a, b, c; $a < b \Leftrightarrow a - c < b - c$

   Thus it follows that $a < b \Leftrightarrow a + (-c) < b + (-c)$

   $\Rightarrow a < b \Leftrightarrow a - c < b - c$ ■

3. For all real numbers a, b, c with $c > 0$; $a < b \Leftrightarrow ac < bc$

Proof.

$\boxed{a < b \Rightarrow ac < bc}$

Let the hypothesis be given. (i.e., let $a < b$ and $c > 0$)

$\Rightarrow b - a$ is positive

$\Rightarrow (b - a)\, c$ is positive because it is the product of two positive real numbers

$\Rightarrow bc - ac$ is positive

$\Rightarrow ac < bc$

i.e., $a < b \Rightarrow ac < bc$

$\boxed{ac < bc \Rightarrow a < b}$

Let the hypothesis be given. (i.e., let $ac < bc$ and $c > 0$)

$\Rightarrow bc - ac$ is positive

$\Rightarrow (bc - ac)\, \frac{1}{c}$ is positive because it is the product of positive real numbers

$\Rightarrow bc\frac{1}{c} - ac\frac{1}{c}$ is positive

$\Rightarrow b\left(c\frac{1}{c}\right) - a\left(c\frac{1}{c}\right)$ is positive

$\Rightarrow b - a$ is positive

$\Rightarrow a < b$

i.e., $ac < bc \Rightarrow a < b$ ■

4. For all real numbers a, b, c with $c > 0$; $a < b \Leftrightarrow \frac{a}{c} < \frac{b}{c}$

Proof.

Let the hypotheses be given.

From problem 3, we have:

For all real numbers a, b, c with $c > 0$; $a < b \Leftrightarrow ac < bc$

Thus, it follows that $a\frac{1}{c} < b\frac{1}{c}$ because $\frac{1}{c}$ is positive

$\Rightarrow \frac{a}{c} < \frac{b}{c}$

i.e., $a < b \Leftrightarrow \frac{a}{c} < \frac{b}{c}$

5. For all real numbers a, b, c with $c < 0$; $a < b \Leftrightarrow ac > bc$

Proof.

$\boxed{a < b \Rightarrow ac > bc}$

Let the hypothesis be given. (i.e., let $a < b$ and $c < 0$)

$\Rightarrow b - a$ is positive

$\Rightarrow (b - a)(-c)$ is positive because it is the product of two positive real numbers

$\Rightarrow -bc + ac$ is positive

$\Rightarrow ac - bc$ is positive

$\Rightarrow ac > bc$

i.e., $a < b \Rightarrow ac > bc$

$\boxed{ac > bc \Rightarrow a < b}$

Let the hypothesis be given. (i.e., let $ac > bc$ and $c < 0$)

$\Rightarrow ac - bc$ is positive

$\Rightarrow bc - ac$ is negative

$\Rightarrow (bc - ac)\frac{1}{c}$ is positive because it is the product of two negative real numbers

$\Rightarrow bc\frac{1}{c} - ac\frac{1}{c}$ is positive

$\Rightarrow b\left(c\frac{1}{c}\right) - a\left(c\frac{1}{c}\right)$ is positive

$\Rightarrow b - a$ is positive

$\Rightarrow a < b$

i.e., $ac > bc \Rightarrow a < b$ ■

6. For all real numbers a, b, c with $c < 0$; $a < b \Leftrightarrow \frac{a}{c} > \frac{b}{c}$

Proof.

Let the hypotheses be given.

From problem 5, we have:

For all real numbers a, b, c with $c < 0$; $a < b \Leftrightarrow ac > bc$

Thus, it follows that $a\frac{1}{c} > b\frac{1}{c}$ because $\frac{1}{c}$ is negative

$\Rightarrow \frac{a}{c} > \frac{b}{c}$

i.e., $a < b \Leftrightarrow \frac{a}{c} < \frac{b}{c}$

7. $0 < r < |x| \Leftrightarrow x < -r$ or $r < x$

Proof.

$\boxed{0 < r < |x| \Rightarrow x < -r \text{ or } r < x}$

Let the hypothesis be given. (i.e., suppose that $0 < r < |x|$)

There are two cases to be considered:

case 1: $x > 0$

$\Rightarrow |x| = x$

Thus, our hypothesis becomes: $0 < r < x$

i.e., $r < x$

case 2: $x < 0$

$\Rightarrow |x| = -x$

Thus, our hypothesis becomes: $0 < r < -x$

$\Rightarrow r < -x$

$\Rightarrow -r > x$

i.e., $x < -r$

Thus, $0 < r < |x| \Rightarrow x < -r$ or $r < x$

$\boxed{x < -r \text{ or } r < x \Rightarrow 0 < r < |x|}$

(Note from the context that $r > 0$.)

Let the hypothesis be given. (i.e., suppose that either $x < -r$ or $r < x$)

If $x < -r$, then $x < -r < 0$

i.e., $x < -r \Rightarrow x < 0$

$\Rightarrow |x| = -x$

$\Rightarrow x = -|x|$

$\Rightarrow -|x| < -r$

$\Rightarrow |x| > r$

If $x \not< -r$, then $r < x$, by hypothesis.

$\Rightarrow 0 < r < x$

i.e., $r < x \Rightarrow 0 < x$

$\Rightarrow |x| = x$

$\Rightarrow r < |x|$

i.e., $x < -r$ or $r < x \Rightarrow 0 < r < |x|$ ■

8. $0 \leq |x| < r \Leftrightarrow -r < x < r$

Proof.

$\boxed{0 \leq |x| < r \Rightarrow -r < x < r}$

Let the hypothesis be given. (i.e., suppose that $0 \leq |x| < r$)

There are two cases to consider:

case 1: $x \geq 0$

$\Rightarrow |x| = x$

$\Rightarrow 0 \leq x < r$ (by hypothesis)

$\Rightarrow -r < 0 \leq x < r$

$\Rightarrow -r < x < r$

case 2: $x < 0$

$\Rightarrow |x| = -x$

$\Rightarrow 0 \leq -x < r$ (by hypothesis)

$\Rightarrow -r < 0 \leq -x < r$

$\Rightarrow -r < -x < r$

multiplying both sides by -1 yields:

$\Rightarrow r > x > -r$

$\Rightarrow -r < x < r$

i.e. $0 \leq |x| < r \Leftrightarrow -r < x < r$

9. Prove or disprove:

   (a) $x \in \mathbf{Q}$ and $y \in \mathbf{Q} \Rightarrow x + y \in \mathbf{Q}$

   This is TRUE

   Proof. Let the hypothesis be given. (i.e., Suppose that $x, y \in \mathbf{Q}$)

   $\Rightarrow \exists m, n, p, q, \in \mathbf{Z}$ with $n, q \neq 0$ such that $x = \frac{m}{n}$ and $y = \frac{p}{q}$

   Observe: $x + y = \frac{m}{n} + \frac{p}{q} = \frac{mq+pn}{nq} \in \mathbf{Q}$

   i.e., $x + y \in \mathbf{Q}$ ■

   Remark 1 *Our proof hinged heavily on the facts that:*

   1. *the product of integers is an integer*
   2. *the product of non-zero real numbers is a non-zero real number*
   3. *the sum of integers is an integer.*

   Hence, $mq, pn,$ and $mq + pn$ are integers, and nq is a non-zero integer.

   Thus, $\frac{mq+pn}{nq}$ is the quotient of two integers with the denominator being non-zero.

(b) $x \in \mathbf{Q}$ and $y \in \mathbf{Q} \Rightarrow x - y \in \mathbf{Q}$

This is TRUE

Proof. Let the hypothesis be given. (i.e., Suppose that $x, y \in \mathbf{Q}$)

Then $\exists\ p, q \in \mathbf{Z}$ with $q \neq 0$ such that $y = \frac{p}{q}$
$\Rightarrow -y = \frac{-p}{q} \in \mathbf{Q}$ as it is also the quotient of integers (non-zero denominator)
$\Rightarrow x - y = x + (-y) \in \mathbf{Q}$, as it is the sum of rational numbers.
i.e., $x - y \in \mathbf{Q}$ ■

(c) $x \in \mathbf{Q}$ and $y \in \mathbf{Q} \Rightarrow xy \in \mathbf{Q}$

This is TRUE

Proof. Let the hypothesis be given. (i.e., Suppose that $x, y \in \mathbf{Q}$)

$\Rightarrow \exists m, n, p, q, \in \mathbf{Z}$ with $n, q \neq 0$ such that $x = \frac{m}{n}$ and $y = \frac{p}{q}$

Observe: $xy = \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq} \in \mathbf{Q}$

i.e., $x \cdot y \in \mathbf{Q}$ ■

Remark 2 *Our proof hinged heavily on the facts that:*

1. *the product of integers is an integer*
2. *the product of non-zero real numbers is a non-zero real number*

Hence, mp and nq are integers, with nq being a non-zero integer.

Thus, $\frac{mp}{nq}$ is the quotient of two integers with the denominator being non-zero.

(d) $x \in \mathbf{Q}$ and $y \in \mathbf{Q} \Rightarrow \frac{x}{y} \in \mathbf{Q}$

This is FALSE

Given any $x \in \mathbf{Q}$ and $y = 0 \in \mathbf{Q}$, $\frac{x}{y}$ is undefined, and therefore $\frac{x}{y} \notin \mathbf{Q}$.

(e) Under what conditions does $x \in \mathbf{Q}$ and $y \in \mathbf{Q} \Rightarrow \frac{x}{y} \in \mathbf{Q}$?

Given $x \in \mathbf{Q}$ and $y \in \mathbf{Q} \setminus \{0\}$, we have $\frac{x}{y} \in \mathbf{Q}$

Proof. Let the hypotheses be given. (i.e. suppose that $x \in \mathbf{Q}$ and $y \in \mathbf{Q} \setminus \{0\}$)

Then $\exists p, q \in \mathbf{Q} \setminus \{0\}$ such that $y = \frac{p}{q}$

$\Rightarrow \frac{1}{y} = \frac{q}{p}$ is also the quotient of non-zero integers, hence rational.

$\Rightarrow \frac{x}{y} = x\frac{1}{y} \in \mathbf{Q}$ as it is the product of rationals.

i.e., $x \in \mathbf{Q}$ and $y \in \mathbf{Q} \setminus \{0\} \Rightarrow \frac{x}{y} \in \mathbf{Q}$ ■

(f) $x \in \mathbf{Q}$ and $y \in \mathbf{Q}^c \Rightarrow x + y \in \mathbf{Q}^c$

This is TRUE

Proof. (by contradiction)

Let the hypothesis be given. (i.e., suppose that $x \in \mathbf{Q}$ and $y \in \mathbf{Q}^c$)

Suppose, for the sake of deriving a contradiction, that $x + y \in \mathbf{Q}$

$\Rightarrow \exists\ z \in \mathbf{Q}\ \ni x + y = z$

$\Rightarrow y = z - x \in \mathbf{Q}$, since $z - x$ is the difference of two rationals.

This contradicts the fact that y is irrational.

Since the assumption that $x + y \in \mathbf{Q}$ leads to a contradiction, it must be false.

Hence, $x + y \in \mathbf{Q}^c$. ■

(g) $x \in \mathbf{Q}^c$ and $y \in \mathbf{Q}^c \Rightarrow x + y \in \mathbf{Q}^c$

This is FALSE. The sum of two irrationals may be either rational or irrational.

Proof. To show that the sum of two irrationals may be rational, consider $x = \sqrt{2}$ and $y = -\sqrt{2}$

Observe: $x, y \in \mathbf{Q}^c$ and yet, $x + y = 0 \in \mathbf{Q}$

To show that the sum of two irrationals may be irrational, consider $x = y = \sqrt{2}$

Observe: $x, y \in \mathbf{Q}^c$ and yet, $x + y = 2\sqrt{2} \in \mathbf{Q}^c$ (because it's the product of a rational and an irrational.) ■

Alternatively:Proof.
To show that the sum of two irrationals may be rational, consider $x = 0.101001000100001\ldots$ and $y = 0.0101101110111110\ldots$

Observe: $x, y \in \mathbf{Q}^c$ and yet, $x + y = 0.11111111111111\ldots \in \mathbf{Q}$

To show that the sum of two irrationals may be irrational, consider $x = y = 0.101001000100001\ldots$

Observe: $x, y \in \mathbf{Q}^c$ and yet, $x + y = 0.202002000200002\ldots \in \mathbf{Q}^c$ ■

(h) $x \in \mathbf{Q}$ and $y \in \mathbf{Q}^c \Rightarrow xy \in \mathbf{Q}^c$

This is TRUE

Proof. (by contradiction)

Let the hypothesis be given. (i.e., suppose that $x \in \mathbf{Q}$ and $y \in \mathbf{Q}^c$)

Suppose, for the sake of deriving a contradiction, that $x \cdot y \in \mathbf{Q}$

$\Rightarrow \exists\, z \in \mathbf{Q} \ni x \cdot y = z$

$\Rightarrow y = \frac{z}{x} \in \mathbf{Q}$, since $\frac{z}{x}$ is the quotient of two rationals (with the denominator being non-zero).

This contradicts the fact that y is irrational.

Since the assumption that $x \cdot y \in \mathbf{Q}$ leads to a contradiction, it must be false.

Hence, $x \cdot y \in \mathbf{Q}^c$. ■

(i) $x \in \mathbf{Q}^c$ and $y \in \mathbf{Q}^c \Rightarrow xy \in \mathbf{Q}^c$

This is FALSE. The product of two irrationals may be either rational or irrational.

Proof. To show that the product of two irrationals may be rational, consider $x = \sqrt{2}$ and $y = \sqrt{2}$

Observe: $x, y \in \mathbf{Q}^c$ and yet, $x \cdot y = 2 \in \mathbf{Q}$

To show that the product of two irrationals may be irrational, consider $x = \sqrt{2}$ and $y = \sqrt{3}$

Observe: $x, y \in \mathbf{Q}^c$ and $x \cdot y = \sqrt{2}\sqrt{3} = \sqrt{6} \in \mathbf{Q}^c$ (because it's the square root of a non-perfect square") ■