Homework - Sec. 2.3, Page 24

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1. If $a|b$, show that $(-a)\,|b$, $a|\,(-b)$, and $(-a)\,|\,(-b)$.

   Let the hypothesis be given (i.e., suppose that $a|b$).

   Then $\exists$ an integer m such that $b = am$.

   $\Rightarrow b = (-a)(-m) \Rightarrow (-a)\,|b$.

   Also, the fact that $b = am$ implies that $(-b) = a(-m)$, hence, $a|\,(-b)$.

   Finally, the fact that $b = am$ implies that $(-b) = (-a)(m)$, hence, $(-a)\,|\,(-b)$. ■

2. Given integers a, b, c, d, verify the following:

   (a) If $a|b.$ then $a|bc$.

   Let the hypothesis be given (i.e., suppose that $a|b$).
   Then $\exists$ an integer m such that $b = am$.
   $\Rightarrow bc = (am)\,c = a\,(mc)$
   i.e., $bc = a\,(mc)$.
   Hence, $a|bc$. ■

   (b) If $a|b$ and $a|c,$ then $a^2|bc$.

   Let the hypothesis be given (i.e., suppose that $a|b$ and $a|c$).
   Then $\exists$ an integer(s) m and n such that $b = am$ and $c = an$.
   Then $bc = (am)(an) = a^2(mn)$.
   i.e., $bc = a^2(mn)$, and hence, $a^2|bc$. ■

   (c) $a|b$ if and only if $ac|bc,$ where $c \neq 0$.

   $\boxed{a|b \Rightarrow ac|bc}$
   Suppose that $a|b$. Then $\exists$ an integer m such that $b = am$.
   Observe: $bc = (am)\,c = (ac)\,m$
   i.e., $bc = (ac)\,m,$ and hence, $ac|bc$.
   $\boxed{ac|bc \Rightarrow a|b}$
   Suppose that $ac|bc$. Then $\exists$ an integer m such that $bc = (ac)\,m$.
   $\Rightarrow bc = (am)\,c$.
   Since $c \neq 0$, we can divide both sides by zero.
   $\Rightarrow b = am$.
   Hence, $a|b$. ■

(d) If $a|b$ and $c|d$, then $ac|bd$

Let the hypothesis be given (i.e., suppose that $a|b$ and $c|d$).
Then $\exists$ an integer(s) m and n such that $b = am$ and $d = cn$.
Then $bd = (am)(cn) = (ac)(mn)$.
i.e., $bd = (ac)(mn)$, and hence, $ac|bd$. ■

3. Prove or disprove: If $a|(b+c)$, then either $a|b$ or $a|c$.

This is false. To show that it's false, we need to offer a counterexample.

Consider: $a = 2$, $b = 3$, and $c = 5$.

$a|(b+c)$, but $a \nmid b$ and $a \nmid c$.

5. Prove that for any integer a, one of the integers, $a, a+2, a+4$ is divisible by 3.

Let a be any integer. Then by the division algorithm, there are exactly three mutually exclusive and mutually exhaustive possibilities — either $a = 3k$, $a = 3k+1$, or $a = 3k+2$.

$\boxed{\text{Case 1: } a = 3k}$

If $a = 3k$, then our assertion is proved.

$\boxed{\text{Case 2: } a = 3k+1}$

If $a = 3k+1$, then $a + 2 = (3k+1) + 2 = 3k + 3 = 3(k+1)$

i.e., $a + 2 = 3(k+1)$. Hence, $3|(a+2)$

$\boxed{\text{Case 2: } a = 3k+2}$

If $a = 3k+2$, then $a + 4 = (3k+2) + 4 = 3k + 6 = 3(k+2)$

i.e., $a + 4 = 3(k+2)$. Hence, $3|(a+4)$

Since all possibilities have been exhausted and since our proposition is true for each possibility, we can say that for any integer a, one of the integers, $a, a+2, a+4$ is divisible by 3. ■

8. ~

   (a) The sum of the squares of two odd integers cannot be a perfect square.

   Two arbitrary odd integers can be represented as $2k+1$ and $2j+1$.
   The sum of their squares is $(2k+1)^2+(2j+1)^2 = 4k^2+4k+1+4j^2+4j+1 = 4k^2+4k+4j^2+4j+2 = 4\left(k^2+k+j^2+j\right)+2 = 4m+2$, where $m = k^2+k+j^2+j$
   The point here, is that the sum of the squares of any two odd numbers is of the form $4m+2$.
   Could this be a perfect square? Well, let's see what form(s) perfect squares can have.
   An even number $2n$, when squared, has the form: $4n^2 = 4m$, where $m = n^2$.
   An odd number $2n+1$, when squared, has the form: $4n^2+4n+1 = 4m+1$, where $m = n^2+n$.
   What we have learned is that perfect squares are either of the form $4m$ or $4m+1$. Perfect squares are *never* of the form $4m+2$, which is the form that the sum of the squares of two odd numbers always has.
   Hence, the sum of the squares of two odd integers cannot be a perfect square. ■

   (b) The product of four consecutive integers is one less than a perfect square.

   We can represent the four consecutive integers as $n, n+1, n+2, n+3$.
   Their product is $n(n+1)(n+2)(n+3) = n^4+6n^3+11n^2+6n$
   This is one less than $n^4+6n^3+11n^2+6n+1 = \left(n^2+3n+1\right)^2$, which is a perfect square. ■

9. Establish that the difference of two consecutive cubes is never divisible by 2.

$\boxed{\text{Case 1: The smaller number is even.}}$

Let a be even. Then $\exists$ a natural number k such that $a = 2k$. This being the case, $a+1 = 2k+1$

The difference of the cubes of these numbers is $(a+1)^3 - a^3 = (2k+1)^3 - (2k)^3 = 12k^2+6k+1 = \underbrace{2\left(6k^2+3k\right)+1}_{2m+1}$, which is odd.

$\boxed{\text{Case 2: The smaller number is odd.}}$

Let a be odd. Then $\exists$ a natural number k such that $a = 2k+1$. This being the case, $a+1 = 2k+2$

The difference of the cubes of these numbers is $(a+1)^3 - a^3 = (2k+2)^3 - (2k+1)^3 = 12k^2+18k+7 = \underbrace{2\left(6k^2+9k+3\right)+1}_{2m+1}$, which is odd.

Since this exhausts all cases, and each case results in the difference of two consecutive cubes being odd, our assertion is proved. ■

11. If a and b are integers, not both of which are zero, verify that:

$$\gcd(a,b) = \gcd(-a,b) = \gcd(a,-b) = \gcd(-a,-b).$$

$d = \gcd(a,b)$

$\Leftrightarrow$ d is the smallest natural number such that $\exists x, y \in \mathbf{Z}$ such that $ax + by = d$

$\Leftrightarrow$ d is the smallest natural number such that$\exists x_1, y \in \mathbf{Z}$ such that

$$\underbrace{(-a)\,x_1}_{x_1 = -x} + by = d \tag{Eq.1}$$

(and consequently $d = \gcd(-a,b)$)

$\Leftrightarrow$ d is the smallest natural number such that $\exists x_1, y_1 \in \mathbf{Z}$ such that

$\underbrace{(-a)\,x_1}_{x_1 = -x} + \underbrace{(-b)\,y_1}_{y_1 = -y} = d$

(and consequently $d = \gcd(-a,-b)$)

$\Leftrightarrow$ d is the smallest natural number such that $\exists x, y_1 \in \mathbf{Z}$ such that $\underbrace{(a)\,x}_{x = x_1} + \underbrace{(-b)\,y_1}_{y_1 = -y} = d.$

(and consequently $d = \gcd(a,-b)$) $\blacksquare$

Remark 1 *In Eq. 1, if d were NOT the smallest natural number such that*

$\underbrace{(-a)\,x_1}_{x_1=-x} + by = d$*, then there must be a smaller natural number c such that*

$\underbrace{(-a)\,x_1}_{x_1=-x} + by = c.$

But this would imply that $\underbrace{ax_1}_{x_1=x} + by = c$, contradicting the assumption that d was the smallest such natural number.

Remark 2 *This was not the solution that I originally had.* ***Julian Allagan*** *showed me this proof and I liked it, so I changed the one that I had.*

18a. **Prove:** The product of any three consecutive integers is divisible by 6.

Let the integers be represented as n, $n+1$, $n+2$.

Claim: At least one of the integers must be even.

By the Division Algorithm, n is either of the form $2k$ or $2k+1$.

If n is of the form $2k$, our claim is proved.

If n is of the form $2k+1$, then $n+1 = (2k+1)+1 = 2(k+1)$, and hence, $n+1$ is even.

End of Claim

Claim: At least one of the integers must be divisible by 3.

By the Division Algorithm, n is either of the form $3k$, $3k+1$, or $3k+2$.

If n is of the form $3k$, then our claim is proved.

If n is of the form $3k+1$, then $n+2 = (3k+1)+2 = 3(k+1)$, and hence, $n+2$ is divisible by 3.

If n is of the form $3k+2$, then $n+1 = (3k+2)+1 = 3(k+1)$, and hence, $n+1$ is divisible by 3.

End of Claim

Thus $2|\,(n)(n+1)(n+2)$ and $3|\,(n)(n+1)(n+2)$.

Since $\gcd(2,3) = 1$, the second corollary to Theorem 2.4 (**Divisibility Theorem 1)** tells us that

$(2 \cdot 3) \mid (n)(n+1)(n+2)$.

i.e., the product of any three consecutive integers is divisible by 6. ■

Remark 3 *The following is an Alternate Proof by* ***Madison Butler.*** *She approaches the problem from a completely different perspective and I really like the proof.*

Recall that $\binom{k+2}{3}$ represents the number of ways in which 3 objects can be selected from a set of $k+2$ objects, disregarding the order of selection. Hence, $\binom{k+2}{3}$ is a natural number.

Observe that $\binom{k+2}{3} = \frac{(k+2)!}{3!((k+2)-3)!} = \frac{(k+2)!}{3!(k-1)!} = \frac{(k+2)(k+1)k(k-1)!}{3!(k-1)!} = \frac{(k+2)(k+1)k}{3!} = \frac{(k+2)(k+1)k}{6}$

i.e., $k(k+1)(k+2)$, where $k, (k+1), (k+2)$ are natural numbers, is divisible by 6.

In the case in which $k, (k+1), (k+2)$ are negative integers, note that:

$k(k+1)(k+2) = (-1)|k||k+1||k+2|$, and that $|k||k+1||k+2|$ is divisible by 6, by our previous

observation.

Hence, $(-1)|k||k+1||k+2| = k(k+1)(k+2)$ is also divisible by 6.

In the case in which at least one, but not all of $k, (k+1), (k+2)$ are negative integers, the product $k(k+1)(k+2) = 0$, and is therefore divisible by 6. ■

18b **Prove:** The product of any four consecutive integers is divisible by 24.

Let the integers be represented as n, $n+1$, $n+2$, and $n+3$.

By our results in part 11a, the product $(n)(n+1)(n+2)(n+3)$ is divisible by 3.

Claim: Our product is divisible by 8.

By the Division Algorithm, n is either of the form $4k$, $4k+1$, $4k+2$, or $4k+3$.

If n is of the form $4k$, then $4k+2 = 2(2k+1)$, and the product

$(n)(n+1)(n+2)(n+3) = (4k)(4k+1)(4k+2)(4k+3) = (4k)(4k+1)2(2k+1)(4k+3)$

$= 8k(4k+1)(2k+1)(4k+3)$. Our claim is proved.

If n is of the form $4k+1$, then $n+1 = (4k+1)+1 = 2(2k+1)$. Furthermore, $n+3 = (4k+1)+3 = 4(k+1)$.

Hence, our product $(n)(n+1)(n+2)(n+3) = (4k+1)(4k+2)(4k+3)(4k+4) =$

$(4k+1)2(2k+1)(4k+3)4(k+1) = 8(4k+1)(2k+1)(4k+3)(k+1)$.

And our claim is proved.

Similar arguments can be used to show that if $n = 4k+2$ or $n = 4k+3$, then $8|(n)(n+1)(n+2)(n+3)$

End of Claim

We have established that $8|(n)(n+1)(n+2)(n+3)$ and that $3|(n)(n+1)(n+2)(n+3)$

Since $\gcd(3,8) = 1$, the second corollary to Theorem 2.4 tells us that

$$(3 \cdot 8) \,|\, (n)(n+1)(n+2)(n+3).$$

i.e., the product of any four consecutive integers is divisible by 24. ■

Remark 4 *The approach used by* ***Madison Butler*** *in part* ***18 a*** *can be used in lieu of the preceding proof.*

18c **Prove:** The product of any five consecutive integers is divisible by 120.

By our results in part 11b, the product $(n)(n+1)(n+2)(n+3)(n+4)$ is divisible by 24.

Since $\gcd(5, 24) = 1,$ it remains to show that $5|(n)(n+1)(n+2)(n+3)(n+4)$.

An argument similar to those of part 11a., can be used to show this.

Hence, the product of five consecutive integers is divisible by 120. ■

Remark 5 *The approach used by* ***Madison Butler*** *in* ***18a*** *can be used in lieu of the preceding proof.*