MTH 1126 Practice Test #5 - Solutions

Fall 2015

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Instructions. In exercises 1 - 9 determine whether the given series converges or diverges.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$

   The series is alternating.

   (a) $\frac{1}{n^2} > \frac{1}{(n+1)^2}$ so $a_n > a_{n+1}$

   (b) $\lim_{n\to\infty}\frac{1}{n^2} = 0$, so $\lim_{n\to\infty} a_n = 0$

   The series converges by the alternating series test.

   i.e., $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ converges by the Alternating Series Test.

2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n)!}$

   The series is alternating.

   (a) $\frac{1}{(2n)!} > \frac{1}{[2(n+1)]!}$ because $2(n+1)! > (2n)!$. So $a_n > a_{n+1}$

   (b) $\lim_{n\to\infty}\frac{1}{(2n)!} = 0$, so $\lim_{n\to\infty} a_n = 0$

   The series converges by the alternating series test.

   i.e., $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n)!}$ converges by the Alternating Series Test.

3. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{n+4}$

   This series is alternating.

   However, $\lim_{n\to\infty}\frac{n+1}{n+4} = 1 \neq 0$. This means that $\lim_{n\to\infty} a_n \neq 0$.

   Therefore, the series diverges.

   i.e., $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{n+4}$ diverges because $a_n \nrightarrow 0$

4. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+2)}$

The series is alternating.

(a) $\frac{1}{\ln(n+2)} > \frac{1}{\ln[(n+1)+2]}$ because $\ln[(n+1)+2] = \ln(n+3) > \ln(n+2)$.

(Because $\ln(x)$ is increasing. We know this beacuse of what the graph of $\ln(x)$ looks like.)

$\Rightarrow a_n > a_{n+1}$

(b) $\lim_{n\to\infty} \frac{1}{\ln(n+2)} = 0$, so $\lim_{n\to\infty} a_n = 0$

Therefore, the series converges by the alternating series test.

i.e., $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+2)}$ converges by the Alternating Series Test.

5. $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

We will get a lot of the factors in numerator and denominator to cancel if we use the Ratio Test, so let's use it.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{[2(n+1)]!}}{\frac{n!}{(2n)!}}\right| = \lim_{n\to\infty} \frac{(n+1)!}{[2(n+1)]!} \cdot \frac{(2n)!}{n!} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot n!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{n!} = \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n+1}{4n^2+6n+2} = 0$$

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. By the Ratio Test, the series converges.

i.e., $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$ converges by the Ratio Test.

6. $\sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!}{e^n}$

Again, if we use the Ratio Test, a lot of factors of a_n and a_{n+1} will cancel.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} \frac{(2(n+1)-1)!}{e^{n+1}}}{(-1)^n \frac{(2n-1)!}{e^n}}\right| = \lim_{n\to\infty} \frac{(2n+1)!}{e^{n+1}} \cdot \frac{e^n}{(2n-1)!}$$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n)(2n-1)!}{e\cdot e^n} \cdot \frac{e^n}{(2n-1)!} = \lim_{n\to\infty} \frac{(2n+1)(2n)}{e} = \infty$$

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$. By the Ratio Test, the series diverges.

i.e., $\sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!}{e^n}$ diverges by the Ratio Test.

7. $\sum_{n=1}^{\infty} n^2 \left(\frac{3}{7}\right)^n$

Since the main feature of a_n is something raised to the n^{th} power, we can get rid of the exponent by using the n^{th} root test, so let's use it!

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{n^2 \left(\frac{3}{7}\right)^n} = \lim_{n\to\infty} \sqrt[n]{n^2} \sqrt[n]{\left(\frac{3}{7}\right)^n} = \lim_{n\to\infty} \sqrt[n]{n^2} \left(\frac{3}{7}\right) = \left(\frac{3}{7}\right) \lim_{n\to\infty} \sqrt[n]{n^2}$$

$$= \left(\frac{3}{7}\right) \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^2 = \underbrace{\left(\frac{3}{7}\right)\left(\lim_{n\to\infty} \sqrt[n]{n}\right)^2}_{\text{Recall: } \lim_{n\to\infty} \sqrt[n]{n}=1} = \frac{3}{7} \cdot 1 = \frac{3}{7}$$

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{3}{7} < 1$, The series converges by the n^{th} root test.

> i.e., $\sum_{n=1}^{\infty} n^2 \left(\frac{3}{7}\right)^n$ converges by the n^{th} Root Test.

8. $\sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{n}\right)^n$

Again, the main feature of a_n is something raised to the n^{th} power, so let's use the n^{th} root test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{2} + \frac{1}{n}\right)^n} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{n}\right) = \frac{1}{2}$$

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2} < 1$, the series converges by the n^{th} root test.

> i.e., $\sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{n}\right)^n$ converges by the n^{th} Root Test.

9. $\sum_{n=1}^{\infty} n^n \left(\frac{3}{5}\right)^n$

Using the n^{th} root test, we have:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{n^n \left(\frac{3}{5}\right)^n} = \lim_{n\to\infty} \sqrt[n]{n^n} \sqrt[n]{\left(\frac{3}{5}\right)^n} = \lim_{n\to\infty} n \left(\frac{3}{5}\right) = \infty$$

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty > 1$, The series diverges by the n^{th} root test.

> i.e., $\sum_{n=1}^{\infty} n^n \left(\frac{3}{5}\right)^n$ diverges by the n^{th} Root Test.

In exercises 10 - 12, determine whether the given series is divergent, conditionally convergent, or absolutely convergent.

10. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$

Consider $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{2^n}{n!}$.

Using the Ratio Test, we have:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2\cdot 2^n}{(n+1)n!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

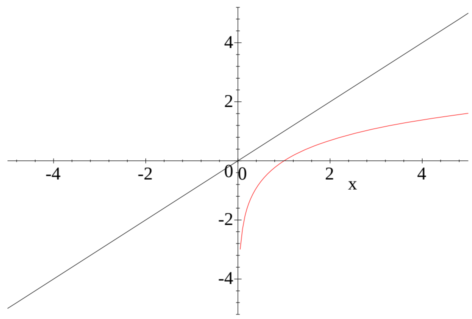
Thus, the series $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$ converges absolutely.

i.e., $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$ converges absolutely.

11. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}$

Consider $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{\ln(n+1)}\right| = \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

It is helpful for us to know that $\ln(x) < x$. This can be seen by looking at the graphs of $y = x$ and $y = \ln(x)$.



$y = x$ and $y = \ln(x)$

Since $\ln(x) < x$, it follows that $\ln(x+1) < x+1$, and hence, $\frac{1}{x+1} < \frac{1}{\ln(x+1)}$.

From this, we see that $\frac{1}{n+1} < \frac{1}{\ln(n+1)}$.

It follows that $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ diverges by direct comparison to $\sum_{n=1}^{\infty} \frac{1}{n+1}$ which is the harmonic series, with first term $\frac{1}{2}$. Therefore, the series does not converge absolutely.

However, the original series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}$ is alternating.

(a) $\frac{1}{\ln(n+1)} > \frac{1}{\ln[(n+1)+1]}$, so $a_n > a_{n+1}$

(b) $\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$, so $\lim_{n\to\infty} a_n = 0$.

Therefore, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}$ converges by the alternating series test, and the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}$ converges conditionally.

i.e., $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}$ converges conditionally.

12. $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n)!}$

Consider $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|(-1)^n \frac{n!}{(2n)!}\right| = \sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

Using the Ratio Test, we have:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{(2(n+1))!}}{\frac{n!}{(2n)!}}\right| = \lim_{n\to\infty} \frac{(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{n!} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)n!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{n!} = \lim_{n\to\infty} \frac{n+1}{4n^2+6n+2} = 0$$

Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$ converges by the Ratio Test. Therefore, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n)!}$ converges absolutely.

i.e., $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n)!}$ converges absolutely.

In problems 13 - 15 simplify (identify) the given expression.

13. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

This should be memorized!

i.e., $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = e^x$

14. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

This should be memorized!

i.e., $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sin(x)$

15. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

This should be memorized!

i.e., $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \cos(x)$

16. Find the Taylor Series for $f(x) = \sin(x)$ centered at $c = \frac{\pi}{4}$.

Use: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-c)^n$ with $c = \frac{\pi}{4}$. First, we compute the first few derivatives of $f(x)$.$f^{(0)}(x) = f(x) = \sin(x) \Rightarrow f^{(0)}\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f^{(1)}(x) = f'(x) = \cos(x) \Rightarrow f^{(1)}\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f^{(2)}(x) = f''(x) = -\sin(x) \Rightarrow f^{(2)}\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{(3)}(x) = f'''(x) = -\cos(x) \Rightarrow f^{(3)}\left(\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{(4)}(x) = \sin(x) \Rightarrow f^{(4)}\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Therefore:

$$f(x) = \sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-c)^n = \frac{\frac{\sqrt{2}}{2}}{0!}\left(x-\frac{\pi}{4}\right)^0 + \frac{\frac{\sqrt{2}}{2}}{1!}\left(x-\frac{\pi}{4}\right)^1 + \frac{-\frac{\sqrt{2}}{2}}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{-\frac{\sqrt{2}}{2}}{3!}\left(x-\frac{\pi}{4}\right)^3$$

$$+\frac{\frac{\sqrt{2}}{2}}{4!}\left(x-\frac{\pi}{4}\right)^4 + \ldots$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x-\frac{\pi}{4}\right)^4 + \ldots$$

i.e., $\sin(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x-\frac{\pi}{4}\right)^4 + \ldots$

17. Find the Taylor Series for $f(x) = \frac{1}{x}$ centered at $c = 2$

First, we compute the first few derivatives of $f(x)$.

$f^{(0)}(x) = f(x) = \frac{1}{x} \Rightarrow f^{(0)}(2) = \frac{1}{2}$

$f^{(1)}(x) = f'(x) = -\frac{1}{x^2} \Rightarrow f^{(1)}(2) = -\frac{1}{4}$

$f^{(2)}(x) = f''(x) = \frac{2}{x^3} \Rightarrow f^{(2)}(2) = \frac{2}{8} = \frac{1}{4}$

$f^{(3)}(x) = f'''(x) = -\frac{6}{x^4} \Rightarrow f^{(3)}(2) = -\frac{6}{16} = -\frac{3}{8}$

$f^{(4)}(x) = \frac{24}{x^5} \Rightarrow f^{(4)}(2) = \frac{24}{32} = \frac{3}{4}$

Therefore,

$$f(x) = \frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-c)^n = \frac{\frac{1}{2}}{0!} + \frac{-\frac{1}{4}}{1!}(x-2)^1 + \frac{\frac{1}{4}}{2!}(x-2)^2 + \frac{-\frac{3}{8}}{3!}(x-2)^3 + \frac{\frac{3}{4}}{4!}(x-2)^4 + \ldots$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-c)^4 + \ldots$$

i.e., $\frac{1}{x} = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-c)^4 + \ldots$

18. Find the Taylor Series for $f(x) = \ln(1+x)$ centered at $c = 0$.

First, we compute the first few derivatives of $f(x)$.

$f^{(0)}(x) = f(x) = \ln(1+x) \Rightarrow f^{(0)}(0) = \ln(1) = 0$

$f^{(1)}(x) = f'(x) = (1+x)^{-1} \Rightarrow f^{(1)}(0) = 1 = 0!$

$f^{(2)}(x) = f''(x) = -(1+x)^{-2} \Rightarrow f^{(2)}(0) = -1 = -1!$

$f^{(3)}(x) = f'''(x) = 2(1+x)^{-3} \Rightarrow f^{(3)}(0) = 2 = 2!$

$f^{(4)}(x) = -2 \cdot 3 \cdot (1+x)^{-4} \Rightarrow f^{(4)}(0) = -2 \cdot 3 = -3!$

Therefore:

$$f(x) = \frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-c)^n = \frac{0}{0!} + \frac{0!}{1!}(x-0)^1 + \frac{-1!}{2!}(x-0)^2 + \frac{2!}{3!}(x-0)^3 + \frac{-3!}{4!}(x-0)^4 + \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n+1}x^n}{n} + \ldots$$

i.e., $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n+1}x^n}{n} + \ldots$

In problems 19 - 20 use a known Taylor Series expansion to derive an expansion for the given function.

19. $f(x) = \frac{1-\cos(x)}{x};\ \ x \neq 0.$

Recall: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

Therefore,

$$1 - \cos(x) = 1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots$$

and consequently:

$$\frac{1-\cos(x)}{x} = \frac{1}{x}\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots\right) = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \ldots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n)!}$$

i.e., $\frac{1-\cos(x)}{x} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n)!} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \ldots$

20. $f(x) = \cos\left(x^2\right)$

Let $z = x^2$
Recall: $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$
Therefore, $\cos\left(x^2\right) = \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots = 1 - \frac{\left(x^2\right)^2}{2!} + \frac{\left(x^2\right)^4}{4!} - \frac{\left(x^2\right)^6}{6!} + \ldots = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots$

i.e., $\cos\left(x^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots$