# Applied Max/Min Exercises - Solutions

FALL 2022

Pat Rossi Name ______________________

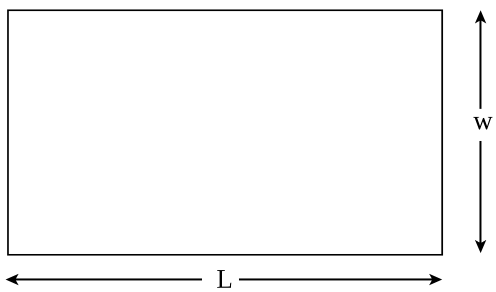
**Instructions.** Use the "5-Step Method" that we've used in class

1. Find the maximum area a rectangle can have if the perimeter is 20 ft.

   i. Determine what it is that we want to maximize/minimize. Give it a name.

   We want to maximize the Area, $A = Lw$ (length times width)

   a. Draw a picture where applicable and/or enlightening.




   ii. Express this quantity as a function of one variable.

   (Oftentimes, we refer to a restriction stated in the problem in order to do this.)

   **Restriction:** Note that the perimeter 20 ft $= 2L + 2w$

   $\Rightarrow 2L = 20 \text{ ft} - 2w$

   $\Rightarrow L = 10\text{ft} - w$

   Plug this into the equation $A = Lw$

   $A = \underbrace{(10\text{ft} - w)}_{L} w$

   $\Rightarrow A(w) = 10\text{ft}w - w^2$

   iii. Determine the restrictions on the independent variable w.

   $w$ is the width of the rectangle. It's possible to make the width $w = 0$ ft, but no smaller.

   It's also possible to make $w = 10$ft, but no larger, as having two sides of length $w = 10$ft exhausts all 20 ft of the perimeter. Thus:

   $0\text{ft} \leq w \leq 10\text{ft}$

iv. Maximize the area, using techniques of calculus.

**Note:** since $A(w)$ is [1]**continuous** (it's a polynomial) on the [2]**closed,** [3]**finite** interval $[0\text{ft}, 10\text{ft}]$, we can use the Absolute Max/Min Value Test.

1. Compute $A'(w)$ and find the critical numbers

   $A'(w) = 10\text{ft} - 2w$

   a. "Type a" $(A'(c) = 0)$

      Set $A'(w) = 10\text{ft} - 2w = 0$

      $\Rightarrow 10\text{ft} - 2w = 0$

      $\Rightarrow 10\text{ft} = 2w$

      $\Rightarrow 5\text{ft} = w$ critical number

   b. "Type b" $(A'(c)$ undefined$)$

      (None)

2. Plug critical numbers and endpoints into the original function

   $A(0\text{ft}) = 10\text{ft}(0\text{ft}) - (0\text{ft})^2 = 0\text{ft}^2$

   $A(5\text{ft}) = 10\text{ft}(5\text{ft}) - (5\text{ft})^2 = 25\text{ft}^2 \longleftarrow$ Absolute Max Area

   $A(10\text{ft}) = 10\text{ft}(10\text{ft}) - (10\text{ft})^2 = 0\text{ft}^2$

v. Make sure that we've answered the original question (solved the original problem)

"Find the maximum area a rectangle can have . . . "

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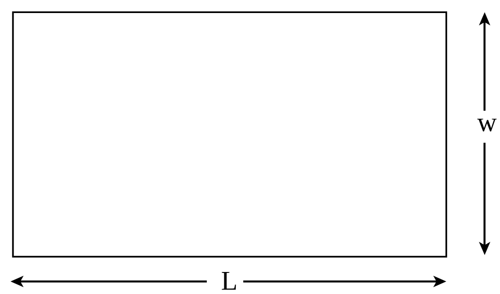
| Maximum Area $= 25\ \text{ft}^2$ |

2. A rectangle has a perimeter of 100 feet. What length and width should it have so that its area is maximum?

   i. Determine what it is that we want to maximize/minimize. Give it a name.

      We want to maximize the Area, $A = Lw$ (length times width)

      a. Draw a picture where applicable and/or enlightening.




   ii. Express this quantity as a function of one variable.

   (Oftentimes, we refer to a restriction stated in the problem in order to do this.)

      **Restriction:** Note that the perimeter 100 ft $= 2L + 2w$

      $\Rightarrow 2L = 100 \text{ ft } - 2w$

      $\Rightarrow L = 50\text{ft } - w$

      Plug this into the equation $A = Lw$

      $A = \underbrace{(50\text{ft} - w)}_{L} w$

      $\Rightarrow A(w) = 50\text{ft}w - w^2$

   iii. Determine the restrictions on the independent variable w.

      $w$ is the width of the rectangle. It's possible to make the width $w = 0$ft, but no smaller.

      It's also possible to make $w = 50$ft, but no larger, as having two sides of length $w = 50$ft accounts for all 100 ft of the perimeter. Thus:

      $0\text{ft} \leq w \leq 50\text{ft}$

iv. Maximize the area, using techniques of calculus.

**Note:** since $A(w)$ is $^1$**continuous** (it's a polynomial) on the $^2$**closed,** $^3$**finite** interval $[0\text{ft}, 50\text{ft}]$, we can use the Absolute Max/Min Value Test.

1. Compute $A'(w)$ and find the critical numbers

   $A'(w) = 50\text{ft} - 2w$

   a. "Type a" $(A'(c) = 0)$

      Set $A'(w) = 50\text{ft} - 2w = 0$

      $\Rightarrow 50\text{ft} - 2w = 0$

      $\Rightarrow 50\text{ft} = 2w$

      $\Rightarrow 25\text{ft} = w$ critical number

   b. "Type b" $(A'(c)$ undefined)

      (None)

2. Plug critical numbers and endpoints into the original function

   $A(0\text{ft}) = 50\text{ft}(0\text{ft}) - (0\text{ft})^2 = 0\text{ft}^2$

   $A(25\text{ft}) = 50\text{ft}(25\text{ft}) - (25\text{ft})^2 = 625\text{ft}^2 \longleftarrow$ Absolute Max Area

   $A(50\text{ft}) = 50\text{ft}(50\text{ft}) - (50\text{ft})^2 = 0\text{ft}^2$

v. Make sure that we've answered the original question (solved the original problem)

"What length and width should it have . . . maximum?"

| |
|---|
| width = 25 ft<br>Length = 25 ft |

3. What positive number x minimizes the sum of x and its reciprocal?

   i. Determine what it is that we want to maximize/minimize. Give it a name.

      We want to maximize the Sum, $S = x + \frac{1}{x}$

      a. Draw a picture where applicable and/or enlightening.

         NA

   ii. Express this quantity as a function of one variable.

      (Already done) $S(x) = x + \frac{1}{x}$

   iii. Determine the restrictions on the independent variable x.

      $x$ is a positive number.

      Therefore, $0 < x$

      Can we "stretch things a little so that $0 \leq x$ ?

      No, because $x = 0$ will make the function $S(x) = x + \frac{1}{x}$ undefined.

      Our restrictions on $x$ are:

      $0 < x$

   iv. Maximize the area, using techniques of calculus.

      **Note:** since $S(x)$ is not restricted to a [2]**closed,** [3]**finite** interval, we can't use the Absolute Max/Min Value Test.

      Well that's just great! Now we'll have to do things the hard way!

      1. Compute $S'(x)$ and find the critical numbers

         For the sake of simplicity, we'll rewrite $S(x)$

         $S(x) = x + \frac{1}{x} = x + x^{-1}$

         i.e., $S(x) = x + x^{-1}$

         $S'(x) = 1 - x^{-2}$

         (Rewrite) $S'(x) = 1 - \frac{1}{x^2}$

         a. "Type a" $(S'(c) = 0)$

            Set $S'(x) = 1 - \frac{1}{x^2} = 0$

            $\Rightarrow 1 = \frac{1}{x^2}$

            $\Rightarrow x^2 = 1$

            $\Rightarrow x = \pm 1$ critical numbers

            Since $x$ must be a positive real number, we discard $x = -1$ as a critical number.

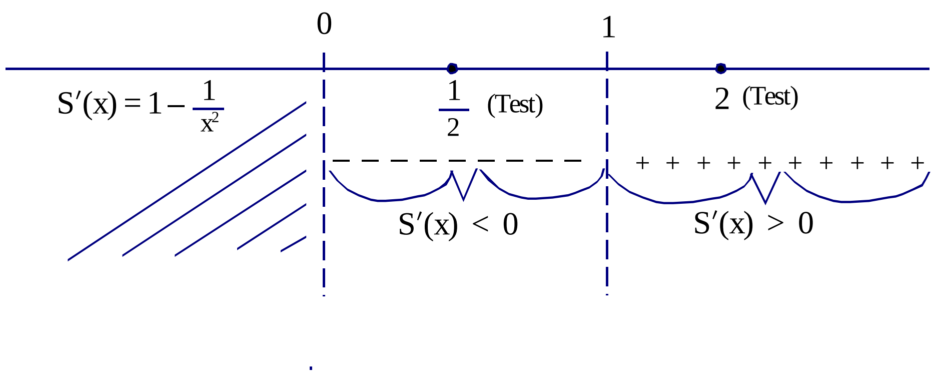
$\Rightarrow x = 1$ critical number

b. "Type b" ($S'(c)$ undefined)

$x = 0$ makes $S'(x)$ undefined, but since $x$ must be a positive real number, we do not accept $x = 0$ as a critical number

2. Draw a "sign graph" of $S'(x)$, using the critical numbers to partition the $x$-axis

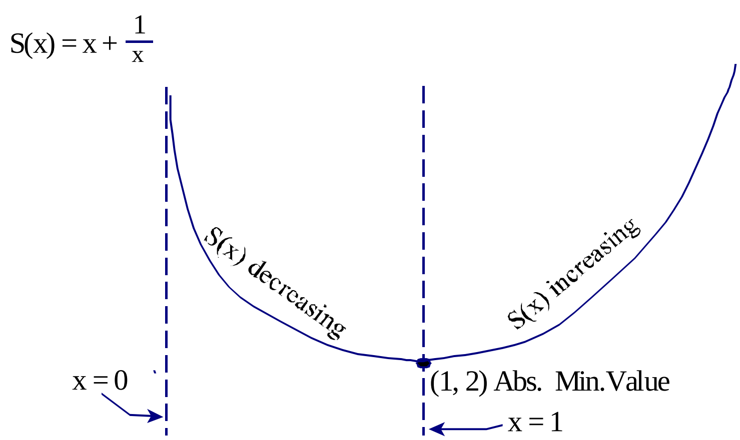
3. From each interval, select a "test value" to plug into $S'(x)$




4. Sketch a rough graph of $S(x)$

$S(x)$ is decreasing on the interval $(0, 1)$ (because $S'(x)$ is negative)

$S(x)$ is increasing on the interval $(1, \infty)$ (because $S'(x)$ is positive)




Note that $S(x)$ attains its absolute minimum value at the point $(1, 2)$.

v. Make sure that we've answered the original question (solved the original problem)

"What positive number x minimizes the sum of x and its reciprocal?"

$$\boxed{x = 1}$$

4. The sum of one number and two times a second number is 24. What numbers should be selected so that their product is as large as possible?

   i. Determine what it is that we want to maximize/minimize. Give it a name.

      We want to maximize the Product, $P = xy$

      a. Draw a picture where applicable and/or enlightening.

         NA

   ii. Express this quantity as a function of one variable.

   (Oftentimes, we refer to a restriction stated in the problem in order to do this.)

      **Restriction:** **"**The sum of one number and two times the second number is 24."

      $\Rightarrow x + 2y = 24$

      $\Rightarrow x = 24 - 2y$

      Plug this into the equation $P = xy$

      $P = \underbrace{(24 - 2y)}_{x} y$

      $\Rightarrow P(y) = 24y - 2y^2$

   iii. Determine the restrictions on the independent variable y.

      Recall that we want the product $P$ "to be as large as possible." Surely $P$ must be positive.

      Since $P = xy$, both x and y must be of the same sign.

      Since $x + y = 24$, both $x$ and $y$ must be positive.

      $x > 0 \Rightarrow \underbrace{24 - 2y}_{=x} > 0 \Rightarrow -2y > -24 \Rightarrow y < 12$

      Combining this with the fact that $y > 0$, we have:

      $0 < y < 12$

      We sure would like to have $0 \leq y \leq 12$ instead. That way, the function $P(y)$ would be restricted to the **closed, finite** interval $[0, 12]$, and we would be able to use the Absolute Max/Min Value Test.

      Note that if $y = 0$, then $P(y) = 24(0) - 2(0)^2 = 0$, which will certainly not be a "false" Abs Max Value.

      Similarly, letting $y = 12$ yields $P(y) = 24(12) - 2(12)^2 = 0$, which also will not be a "false" Abs Max Value.

      Hence, we'll let our restrictions on $y$ be:

      $0 \leq y \leq 12$

iv. Maximize the area, using techniques of calculus.

**Note:** since $P(y)$ is [1]**continuous** (it's a polynomial) on the [2]**closed,** [3]**finite** interval $[0, 12]$, we can use the Absolute Max/Min Value Test.

1. Compute $P'(y)$ and find the critical numbers

   $P'(y) = 24 - 4y$

   a. "Type a" $(P'(c) = 0)$

      Set $P'(y) = 24 - 4y = 0$

      $\Rightarrow -4y = -24$

      $\Rightarrow y = 6$ critical number

   b. "Type b" $(P'(y)$ undefined$)$

      (None)

2. Plug critical numbers and endpoints into the original function

   $P(0) = 24(0) - 2(0)^2 = 0$

   $P(6) = 24(6) - 2(6)^2 = 72 \longleftarrow$ Absolute Max Value

   $P(12) = 24(12) - 2(12)^2 = 0$

v. Make sure that we've answered the original question (solved the original problem)

"What numbers should be selected so that their product is as large as possible?"

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| $y = 6$ |
| $x = 24 - 2y = 12$ |

5. Find two positive numbers whose sum is 110 and whose product is a maximum.

   i. Determine what it is that we want to maximize/minimize. Give it a name.

      We want to maximize the Product, $P = xy$

      a. Draw a picture where applicable and/or enlightening.

         NA

   ii. Express this quantity as a function of one variable.

   (Oftentimes, we refer to a restriction stated in the problem in order to do this.)

      **Restriction: ". . .** two positive numbers whose sum is 110."

      $\Rightarrow x + y = 110$

      $\Rightarrow y = 110 - x$

      Plug this into the equation $P = xy$

      $P = \underbrace{(110 - x)}_{y}x$

      $\Rightarrow P(x) = 110x - x^2$

   iii. Determine the restrictions on the independent variable x.

      Recall that x and y must be positive.

      $\Rightarrow x > 0$

      and $y = 110 - x > 0 \Rightarrow -x > -110 \Rightarrow x < 110$

      i.e., $0 < x < 110$

      We sure would like to have $0 \leq x \leq 110$ instead. That way, the function $P(x)$ would be restricted to the **closed, finite** interval $[0, 110]$, and we would be able to use the Absolute Max/Min Value Test.

      Note that if $x = 0$, then $P(x) = 110(0) - (0)^2 = 0$, which will certainly not be a "false" Abs Max Value.

      Similarly, letting $x = 110$ yields $P(x) = 110(110) - (110)^2 = 0$, which also will not be a "false" Abs Max Value.

      Hence, we'll let our restrictions on $x$ be:

      $0 \leq x \leq 110$

iv. Maximize the area, using techniques of calculus.

**Note:** since $P(x)$ is [1]**continuous** (it's a polynomial) on the [2]**closed,** [3]**finite** interval $[0, 110]$, we can use the Absolute Max/Min Value Test.

1. Compute $P'(x)$ and find the critical numbers

   $P'(x) = 110 - 2x$

   a. "Type a" ($P'(c) = 0$)

      Set $P'(x) = 110 - 2x = 0$

      $\Rightarrow -2x = -110$

      $\Rightarrow x = 55$ critical number

   b. "Type b" ($P'(x)$ undefined)

      (None)

2. Plug critical numbers and endpoints into the original function

   $P(0) = 110(0) - (0)^2 = 0$

   $P(55) = 110(55) - (55)^2 = 3025 \longleftarrow$ Absolute Max Value

   $P(110) = 110(110) - (110)^2 = 0$

v. Make sure that we've answered the original question (solved the original problem)

" . . . two positive numbers whose . . . whose product is a maximum."

$x = 55$

$y = 110 - x = 55$

6. Find two positive numbers whose product is 192 and whose sum is a minimum.

   i. Determine what it is that we want to maximize/minimize. Give it a name.

      We want to maximize the Sum, $S = x + y$

      a. Draw a picture where applicable and/or enlightening.

         NA

   ii. Express this quantity as a function of one variable.

   (Oftentimes, we refer to a restriction stated in the problem in order to do this.)

      **Restriction:** "Two positive numbers whose product is 192."

      $\Rightarrow xy = 192$

      $\Rightarrow y = \frac{192}{x}$

      $S(x) = x + y = x + \frac{192}{x}$

      i.e., $S(x) = x + \frac{192}{x}$

   iii. Determine the restrictions on the independent variable x.

      $x$ is a positive number.

      Therefore, $0 < x$

      Can we "stretch things a little so that $0 \leq x$ ?

      No, because $x = 0$ will make the function $S(x) = x + \frac{192}{x}$ undefined.

      Our restrictions on $x$ are:

      $0 < x$

   iv. Maximize the area, using techniques of calculus.

      **Note:** since $S(x)$ is not restricted to a [2]**closed,** [3]**finite** interval, we can't use the Absolute Max/Min Value Test.

      Oh No - Not Again! We'll have to do things the hard way!

      1. Compute $S'(x)$ and find the critical numbers

         For the sake of simplicity, we'll rewrite $S(x)$

         $S(x) = x + \frac{192}{x} = x + 192x^{-1}$

         i.e., $S(x) = x + 192x^{-1}$

         $S'(x) = 1 - 192x^{-2}$

         (Rewrite) $S'(x) = 1 - \frac{192}{x^2}$

         a. "Type a" $(S'(c) = 0)$

            Set $S'(x) = 1 - \frac{192}{x^2} = 0$

$\Rightarrow 1 = \frac{192}{x^2}$

$\Rightarrow x^2 = 192$

$\Rightarrow x = \pm\sqrt{192}$ critical numbers

Since $x$ must be a positive real number, we discard $x = -\sqrt{192}$ as a critical number.

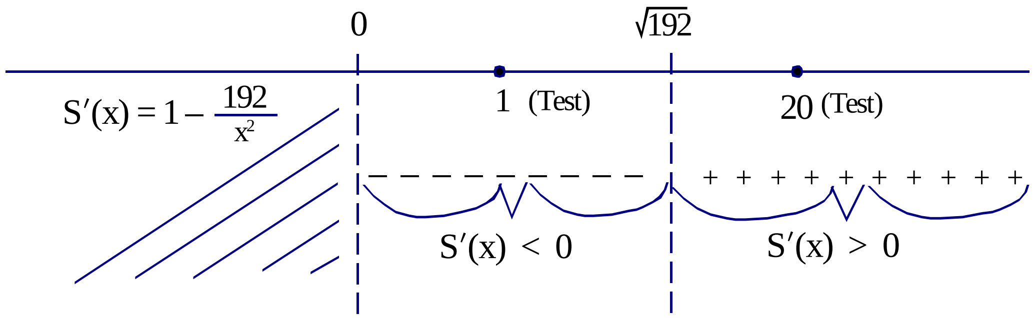
$\Rightarrow x = \sqrt{192}$ "Type a" critical numbers

b. "Type b" ($S'(c)$ undefined)

$x = 0$ makes $S'(x)$ undefined, but since $x$ must be a positive real number, we do not accept $x = 0$ as a critical number

2. Draw a "sign graph" of $S'(x)$, using the critical numbers to partition the $x$-axis

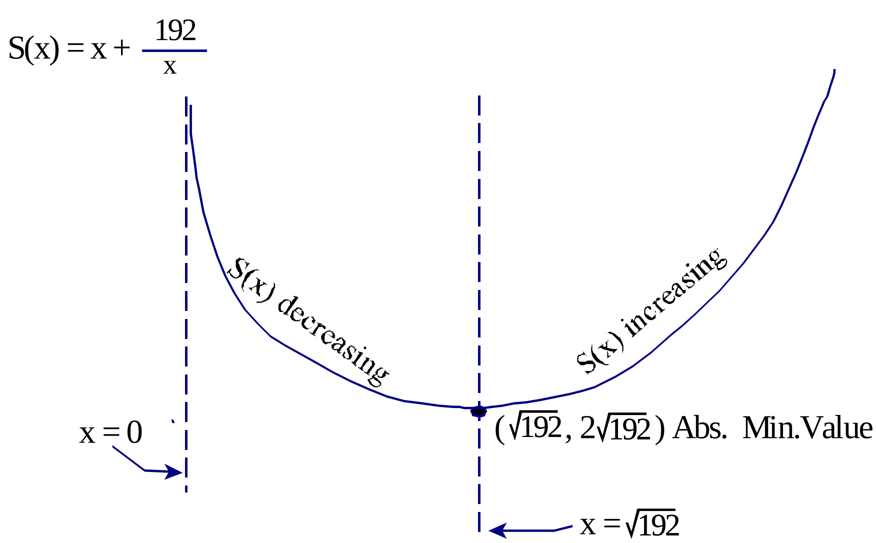
3. From each interval, select a "test value" to plug into $S'(x)$




4. Sketch a rough graph of $S(x)$

$S(x)$ is decreasing on the interval $(0, \sqrt{192})$ (because $S'(x)$ is negative)

$S(x)$ is increasing on the interval $(\sqrt{192}, \infty)$ (because $S'(x)$ is positive)




Note that $S(x)$ attains its absolute minimum value at the point $(\sqrt{192}, 2\sqrt{192})$.

v. Make sure that we've answered the original question (solved the original problem)

"Find two positive numbers whose product is 192 and whose sum is a minimum."

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| $x = \sqrt{192}$ and $y = \frac{192}{x} = \frac{192}{\sqrt{192}} = \sqrt{192}$ |
| |
| i.e. $x = \sqrt{192}$ and $y = \sqrt{192}$ |
| |

7. Find two positive numbers $x$ and $y$ such that $x+y=6$ and $xy^2$ is as large as possible.

i. Determine what it is that we want to maximize/minimize. Give it a name.

We want to maximize the Product, $P = xy^2$

a. Draw a picture where applicable and/or enlightening.

NA

ii. Express this quantity as a function of one variable.

(Oftentimes, we refer to a restriction stated in the problem in order to do this.)

**Restriction: ". . .** two positive numbers $x$ and $y$ such that $x + y = 6$."

$\Rightarrow y = 6 - x$

Plug this into the equation $P = xy^2$

$P = x\underbrace{(6-x)^2}_{y^2}$

$\Rightarrow P(x) = x(6-x)^2 = x(x^2 - 12x + 36) = x^3 - 12x^2 + 36x$

i.e., $P(x) = x^3 - 12x^2 + 36x$

iii. Determine the restrictions on the independent variable x.

Recall that x and y must be positive.

$\Rightarrow x > 0$

and $y = 6 - x > 0 \Rightarrow -x > -6 \Rightarrow x < 6$

i.e., $0 < x < 6$

We sure would like to have $0 \leq x \leq 6$ instead. That way, the function $P(x)$ would be restricted to the **closed, finite** interval $[0, 6]$, and we would be able to use the Absolute Max/Min Value Test.

Note that if $x = 0$, then $P(x) = (0)^3 - 12(0)^2 + 36(0) = 0$, which will certainly not be a "false" Abs Max Value.

Similarly, letting $x = 6$ yields $P(x) = (6)^3 - 12(6)^2 + 36(6) = 0$, which also will not be a "false" Abs Max Value.

Hence, we'll let our restrictions on $x$ be:

$0 \leq x \leq 6$

iv. Maximize the area, using techniques of calculus.

**Note:** since $P(x)$ is [1]**continuous** (it's a polynomial) on the [2]**closed,** [3]**finite** interval $[0, 6]$, we can use the Absolute Max/Min Value Test.

1. Compute $P'(x)$ and find the critical numbers

   $P'(x) = 3x^2 - 24x + 36$

   a. "Type a" $(P'(c) = 0)$

      Set $P'(x) = 3x^2 - 24x + 36 = 0$

      $\Rightarrow 3x^2 - 24x + 36 = 0$

      $\Rightarrow x^2 - 8x + 12 = 0$

      $\Rightarrow (x - 2)(x - 6) = 0$

      $\Rightarrow x = 2, 6$ critical numbers

   b. "Type b" $(P'(x)$ undefined)

      (None)

2. Plug critical numbers and endpoints into the original function

   $P(0) = (0)^3 - 12(0)^2 + 36(0) = 0$

   $P(2) = (2)^3 - 12(2)^2 + 36(2) = 32 \longleftarrow$ Absolute Max Value

   $P(6) = (6)^3 - 12(6)^2 + 36(6) = 0$

v. Make sure that we've answered the original question (solved the original problem)

" . . . two positive numbers $x$ and $y$ such that . . . $xy^2$ is as large as possible."

| |
|---|
| $x = 2$ |
| $y = 6 - x = 4$ |

8. Find two positive numbers such that their product is 36 and the sum of their cubes is a minimum.

i. Determine what it is that we want to maximize/minimize. Give it a name.

We want to maximize the Sum (of the cubes), $S = x^3 + y^3$

a. Draw a picture where applicable and/or enlightening.

NA

ii. Express this quantity as a function of one variable.

(Oftentimes, we refer to a restriction stated in the problem in order to do this.)

**Restriction:** "Two positive numbers such that their product is 36."

$\Rightarrow xy = 36$

$\Rightarrow y = \frac{36}{x}$

$S(x) = x^3 + y^3 = x^3 + \left(\frac{36}{x}\right)^3$

i.e., $S(x) = x^3 + \frac{36^3}{x^3}$

iii. Determine the restrictions on the independent variable x.

$x$ is a positive number.

Therefore, $0 < x$

Can we "stretch things a little so that $0 \leq x$ ?

No, because $x = 0$ will make the function $S(x) = x^3 + \frac{36^3}{x^3}$ undefined.

Our restrictions on $x$ are:

$0 < x$

iv. Maximize the area, using techniques of calculus.

**Note:** since $S(x)$ is not restricted to a [2]**closed,** [3]**finite** interval, we can't use the Absolute Max/Min Value Test.

Oh No - Not Again! We'll have to do things the hard way!

1. Compute $S'(x)$ and find the critical numbers

For the sake of simplicity, we'll rewrite $S(x)$

$S(x) = x^3 + \frac{36^3}{x^3} = x^3 + 36^3x^{-3}$

i.e., $S(x) = x^3 + 36^3x^{-3}$

$S'(x) = 3x^2 - 3 \cdot 36^3x^{-4}$

(Rewrite) $S'(x) = 3x^2 - \frac{3 \cdot 36^3}{x^4}$

a. "Type a" $(S'(c) = 0)$

Set $S'(x) = 3x^2 - \frac{3 \cdot 36^3}{x^4} = 0$

$\Rightarrow 3x^2 = \frac{3 \cdot 36^3}{x^4}$

$\Rightarrow 3x^6 = 3 \cdot 36^3$

$\Rightarrow x^6 = 36^3$

But $36^3 = (6^2)^3 = 6^{2 \cdot 3} = 6^6$

$\Rightarrow x^6 = 6^6$

$\Rightarrow x = \pm 6$ critical numbers

Since $x$ must be a positive real number, we discard $x = -6$ as a critical number.

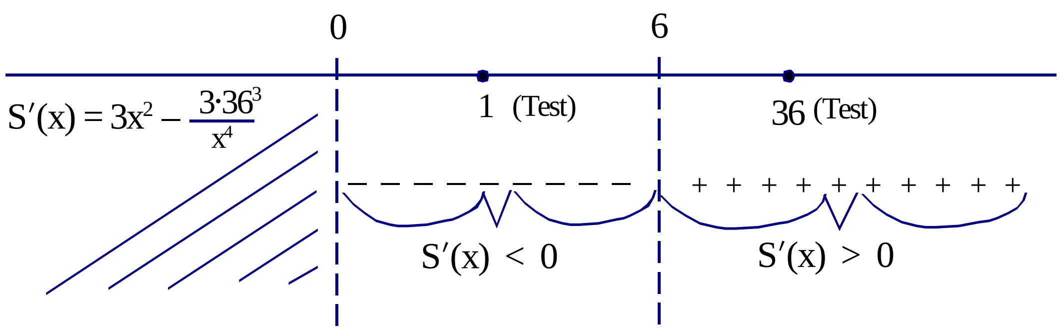
$\Rightarrow x = 6$ "Type a" critical number

b. "Type b" ($S'(c)$ undefined)

$x = 0$ makes $S'(x)$ undefined, but since $x$ must be a positive real number, we do not accept $x = 0$ as a critical number

2. Draw a "sign graph" of $S'(x)$, using the critical numbers to partition the $x$-axis

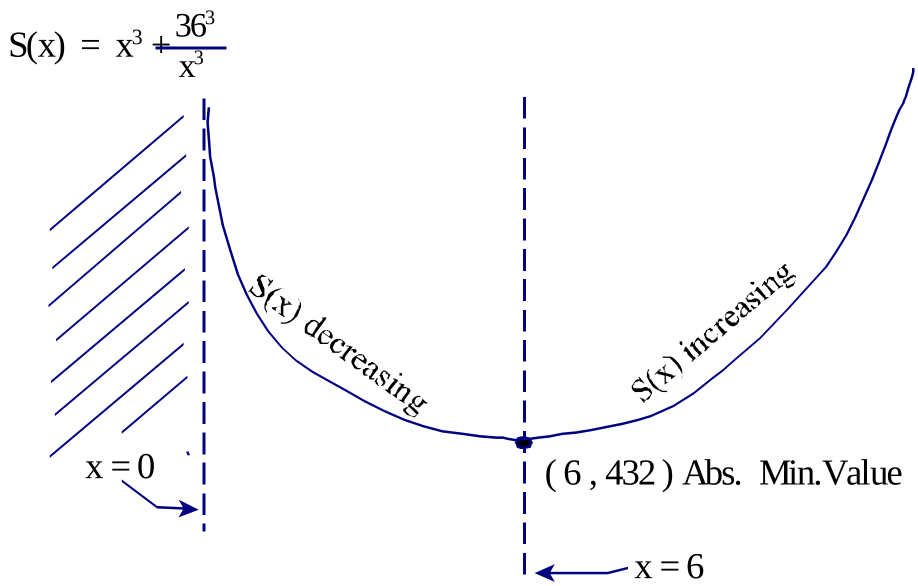
3. From each interval, select a "test value" to plug into $S'(x)$




4. Sketch a rough graph of $S(x)$

$S(x)$ is decreasing on the interval $(0, 6)$ (because $S'(x)$ is negative)

$S(x)$ is increasing on the interval $(6, \infty)$ (because $S'(x)$ is positive)




Note that $S(x)$ attains its absolute minimum value at the point $(6, 432)$.

v. Make sure that we've answered the original question (solved the original problem)

"Find two positive numbers such that their product is 36 and the sum of their cubes is a minimum."

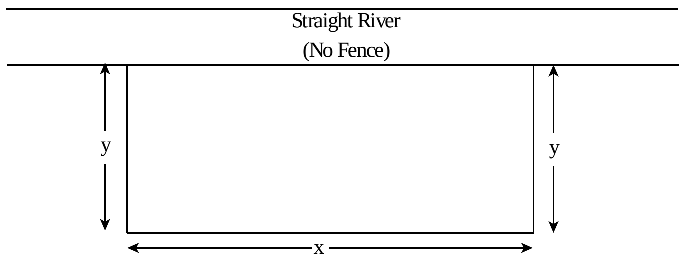
| |
|---|
| $x = 6$ and $y = \frac{36}{x} = \frac{36}{6} = 6$ |
| i.e. $x = 6$ and $y = 6$ |

9. A farmer has 1000 feet of fencing with which to enclose three sides of a rectangular pasture; a straight river will form the fourth side of the pasture. Find the dimensions of the pasture of largest area that the farmer can enclose with the fence.

   i. Determine what it is that we want to maximize/minimize. Give it a name.

      We want to maximize the Area, $A = xy$

      a. Draw a picture where applicable and/or enlightening.




   ii. Express this quantity as a function of one variable.

   (Oftentimes, we refer to a restriction stated in the problem.)

      **Restriction: ". . .** the farmer has 1000 feet of fencing."

      $\Rightarrow x + 2y = 1000\text{ft}$

      $\Rightarrow x = 1000\text{ft} - 2y$

      Plug this into the equation $A = xy$

      $A = \underbrace{(1000\text{ft} - 2y)}_{x} y = 1000\text{ft}\ y - 2y^2$

      $\Rightarrow A(y) = 1000\text{ft}\ y - 2y^2$

   iii. Determine the restrictions on the independent variable y.

      Since $y$ is the width, $y$ can be as small as $= 0\text{ft}$

      Since there are two sides of length $y$, $y$ can be as large as 500 ft, but no larger. If $y = 500$ ft, this will exhaust all 1000 ft of the fencing.

      Hence, our restrictions on y are: $0\text{ft} \leq y \leq 500\text{ft}$

iv. Maximize the area, using techniques of calculus.

**Note:** since $A(y)$ is [1]**continuous** (it's a polynomial) on the [2]**closed,** [3]**finite** interval $[0\text{ft}, 500\text{ft}]$, we can use the Absolute Max/Min Value Test.

1. Compute $A'(y)$ and find the critical numbers

   $A'(y) = 1000\text{ft} - 4y$

   a. "Type a" $(A'(c) = 0)$

      Set $A'(y) = 1000\text{ft} - 4y = 0$

      $\Rightarrow -4y = -1000\text{ft}$

      $\Rightarrow y = 250\text{ft}$ critical number

   b. "Type b" ($A'(y)$ undefined)

      (None)

2. Plug critical numbers and endpoints into the original function

   $A(0\text{ft}) = 1000\text{ft}\ (0\text{ft}) - 2(0\text{ft})^2 = 0\text{ft}^2$

   $A(250\text{ft}) = 1000\text{ft}\ (250\text{ft}) - 2(250\text{ft})^2 = 125,000\text{ft}^2 \longleftarrow$ Absolute Max Value

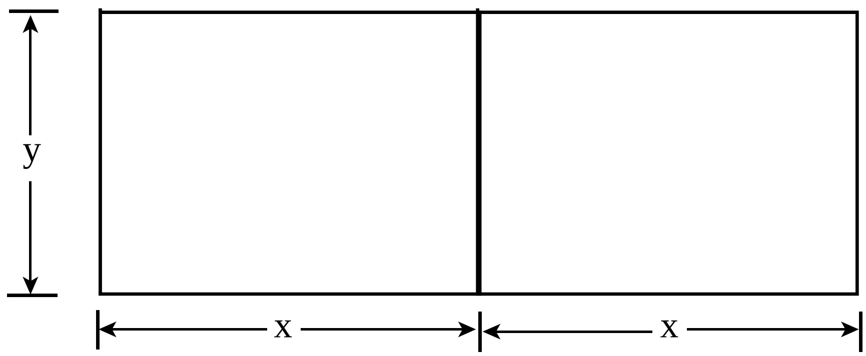
   $A(500\text{ft}) = 1000\text{ft}\ (500\text{ft}) - 2(500\text{ft})^2 = 0\text{ft}^2$

v. Make sure that we've answered the original question (solved the original problem)

"Find the dimensions of the pasture of largest area . . . "

| |
|---|
| $y = 250\text{ft}$ |
| $x = 1000\text{ft} - 2y = 500\text{ft}$ |

10. A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals, as shown in the figure below. (Some of the fencing will be used to form the partition between the two pens.) What dimensions should be used so that the enclosed area will be maximum?




i. Determine what it is that we want to maximize/minimize. Give it a name.

We want to maximize the Area, $A = 2xy$

a. Draw a picture where applicable and/or enlightening.

(None)

ii. Express this quantity as a function of one variable.

(Oftentimes, we refer to a restriction stated in the problem.)

**Restriction: ". . .** a rancher has 200 feet of fencing."

Note that the pen consists of four pieces of length $x$ and three pieces of length $y$.

$\Rightarrow 4x + 3y = 200\text{ft}$

$\Rightarrow 4x = 200\text{ft } - 3y$

$\Rightarrow 2x = 100\text{ft } - \frac{3}{2}y$

Plug this into the equation $A = 2xy$

$$A = \underbrace{\left(100\text{ ft} - \frac{3}{2}y\right)}_{2x} y = 100\text{ft } y - \tfrac{3}{2}y^2$$

$\Rightarrow A(y) = 100\text{ft } y - \frac{3}{2}y^2$

iii. Determine the restrictions on the independent variable y.

Since $y$ is the width, $y$ can be as small as 0ft

Since there are **three** sides of length $y$, $y$ can be as large as $\frac{200}{3}$ ft, but no larger. If $y = \frac{200}{3}$ ft, this will exhaust all 200 ft of the fencing.

Hence, our restrictions on y are: $0\text{ft} \leq y \leq \frac{200}{3}\text{ft}$

iv. Maximize the area, using techniques of calculus.

**Note:** since $A(y)$ is [1]**continuous** (it's a polynomial) on the [2]**closed,** [3]**finite** interval $\left[0\text{ft}, \frac{200}{3}\text{ft}\right]$, we can use the Absolute Max/Min Value Test.

1. Compute $A'(y)$ and find the critical numbers

   $A'(y) = 100\text{ft} - 3y$

   a. "Type a" $(A'(c) = 0)$

      Set $A'(y) = 100\text{ft} - 3y = 0$

      $\Rightarrow -3y = -100\text{ft}$

      $\Rightarrow y = \frac{100}{3}\text{ft}$ critical number

   b. "Type b" ($A'(y)$ undefined)

      (None)

2. Plug critical numbers and endpoints into the original function

   $A(0\text{ft}) = 100\text{ft}\ (0\text{ft}) - \frac{3}{2}(0\text{ft})^2 = 0\text{ft}^2$

   $A\left(\frac{100}{3}\text{ft}\right) = 100\text{ft}\ \left(\frac{100}{3}\text{ft}\right) - \frac{3}{2}\left(\frac{100}{3}\text{ft}\right)^2 = \frac{5000}{3}\text{ft}^2 \longleftarrow$ Absolute Max Value

   $A\left(\frac{200}{3}\text{ft}\right) = 100\text{ft}\ \left(\frac{200}{3}\text{ft}\right) - \frac{3}{2}\left(\frac{200}{3}\text{ft}\right)^2 = 0\text{ft}^2$

v. Make sure that we've answered the original question (solved the original problem)

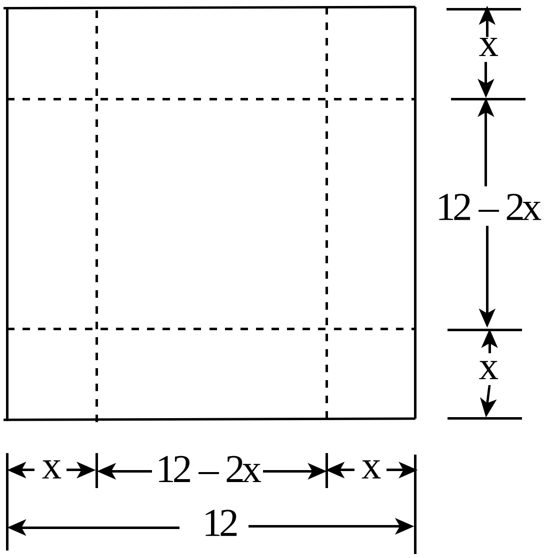
"What dimensions should be used . . . maximum? "

Width $y = \frac{100}{3}\text{ft}$

Length $2x = 100\text{ft} - \frac{3}{2}y = 100ft - \frac{3}{2}\left(\frac{100}{3}\text{ft}\right) = 50\text{ft}$

11. A cardboard box of 108 $cm^3$ volume with a square base and open top is to be constructed. Find the minimum area of cardboard needed. (Neglect material wasted in construction.)

12. An open box is to be made from a square piece of material, 12 inches on a side, by cutting equal squares from each corner and turning up the sides, as shown in the picture below. Find the volume of the largest box that can be made in this manner.

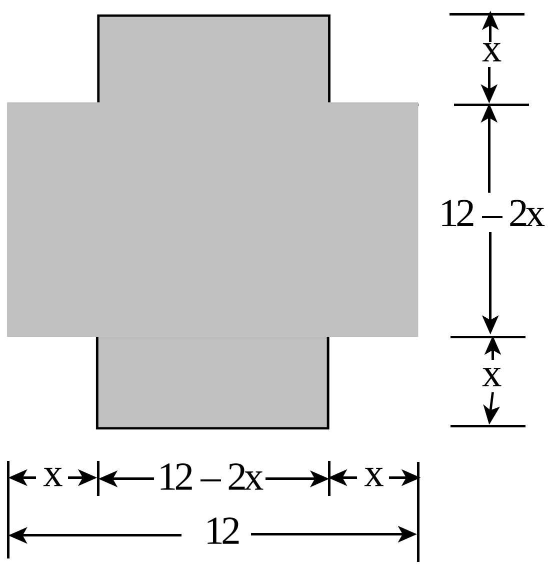



i. Determine what it is that we want to maximize/minimize. Give it a name.

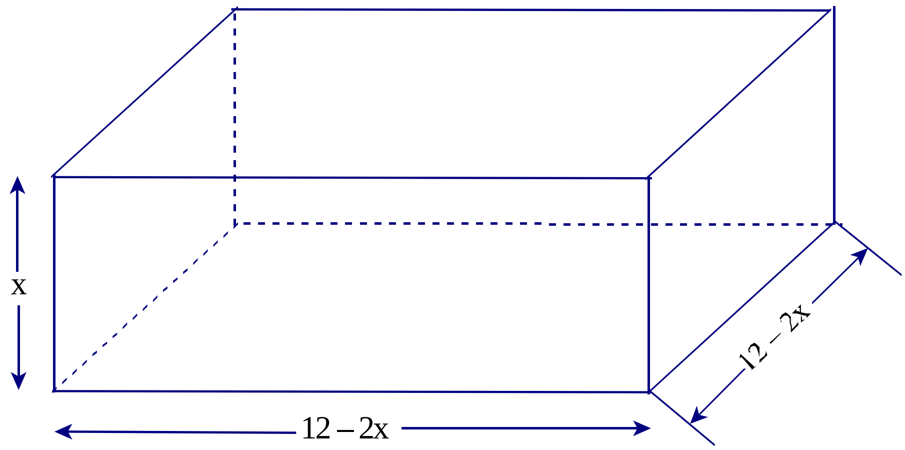
We want to maximize the Volume, $V$

a. Draw a picture where applicable and/or enlightening.

Here, I think that we should draw a number of pictures.




This how the sheet of cardboard will look after we cut out the corners. (above)




This how the box will look after we fold up the corners. (above)

ii. Express this quantity as a function of one variable.

(Oftentimes, we refer to a restriction stated in the problem.)

**From the last sketch, we can see that:**

Length = l = 12 $-2x$

Width = w = 12 $-2x$

Height = h = $x$

Hence, Volume $V = lwh = (12-2x)(12-2x)\,x = 4x^3 - 48x^2 + 144x$

i.e., $V = 4x^3 - 48x^2 + 144x$

iii. Determine the restrictions on the independent variable x.

Since $x$ is the height, $x$ can be as small as 0

Since $x$ is the length and width of the squares that are cut out of the cardboard, $x$ can be equal to 6 inches, but no larger. (Otherwise, the squares that are cut out would overlap.)

Hence, our restrictions on x are: $0 \le x \le 6$

iv. Maximize the area, using techniques of calculus.

**Note:** since $V(x)$ is [1]**continuous** (it's a polynomial) on the [2]**closed,** [3]**finite** interval $[0, 0]$, we can use the Absolute Max/Min Value Test.

1. Compute $V'(x)$ and find the critical numbers

   $V'(x) = 12x^2 - 96x + 144$

   a. "Type a" ($V'(c) = 0$)

      Set $V'(x) = 12x^2 - 96x + 144 = 0$

      $\Rightarrow 12x^2 - 96x + 144 = 0$

      $\Rightarrow x^2 - 8x + 12 = 0$

      $\Rightarrow (x-2)(x-6) = 0$

      $\Rightarrow x = 2$; $x = 6$ critical numbers

   b. "Type b" ($V'(x)$ undefined)

      (None)

2. Plug critical numbers and endpoints into the original function

   $V(0) = 4(0)^3 - 48(0)^2 + 144(0) = 0$

   $V(2) = 4(2)^3 - 48(2)^2 + 144(2) = 128 \longleftarrow$ Absolute Max Value

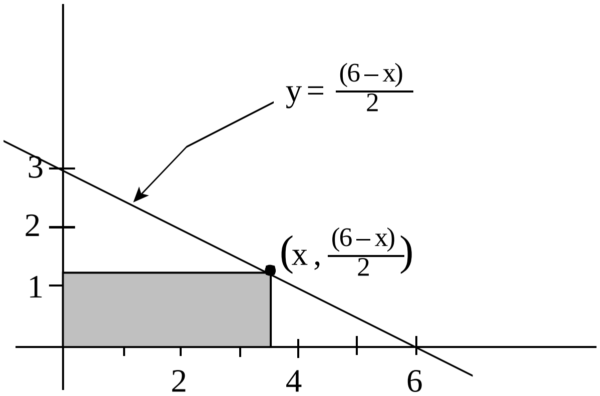
   $V(6) = 4(6)^3 - 48(6)^2 + 144(6) = 0$

v. Make sure that we've answered the original question (solved the original problem)

"Find the volume of the largest box"

| |
|---|
| Volume = 128 |

13. A rectangle is bounded by the x and y axes and the graph of $y = \frac{(6-x)}{2}$ as shown in the picture below. What length and width should the rectangle have so that its area is a maximum?




1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = xy$

   Draw a picture where relevant.

   (Done)

2. Express $A$ as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that the point $(x, y)$ must be on the graph of $y = \frac{(6-x)}{2}$.

   Hence, the $y$-coordinate of the point $(x, y)$ is $y = \frac{(6-x)}{2}$.

   Plug this into the equation $A = xy$

   $\Rightarrow A(x) = x\frac{(6-x)}{2} = 3x - \frac{1}{2}x^2$

   i.e., $A(x) = 3x - \frac{1}{2}x^2$

3. Determine the restrictions on the independent variable $x$.

   From the picture, $0 \leq x \leq 6$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0, 6]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = 3 - x$

a. "Type a" $(f'(c) = 0)$

$\Rightarrow A'(x) = 3 - x = 0$

$\Rightarrow 3 - x = 0$

$\Rightarrow x = 3$ is a critical number

b. "Type b" ($f'(c)$ is undefined)

Look for $x$-values that cause division by zero in $f'(x)$

(None)

2. Plug critical numbers and endpoints into the *original* function.

$A(0) = 3(0) - \frac{1}{2}(0)^2 = 0$

$A(3) = 3(3) - \frac{1}{2}(3)^2 = \frac{9}{2} \longleftarrow$ Abs Max Value

$A(6) = 3(6) - \frac{1}{2}(6)^2 = 0$

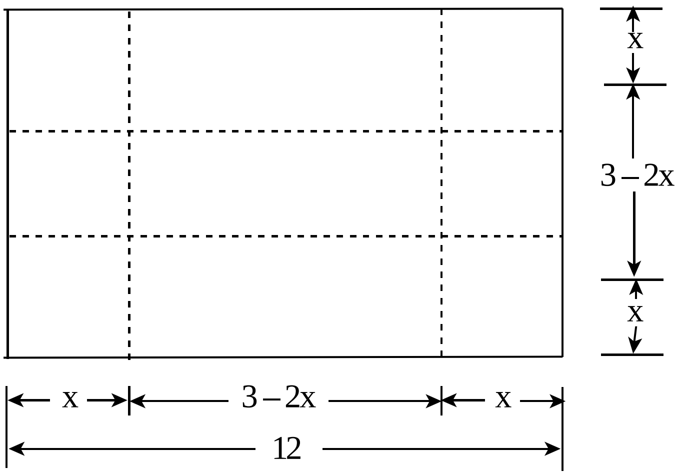
5. Make sure that we've answered the original question.

"What length and width should the rectangle have . . . maximum?"

1. Length $x = 3$

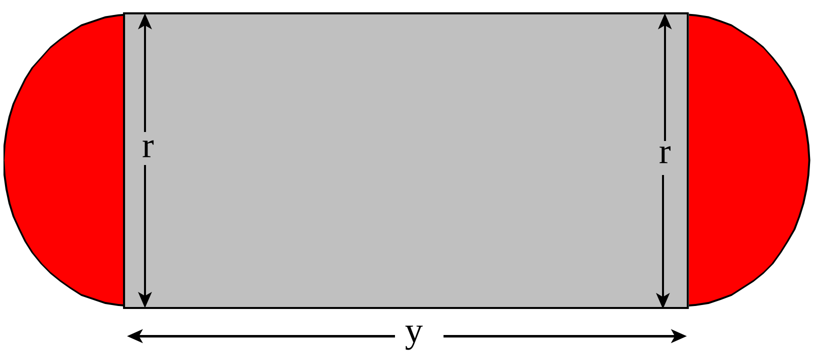
   Width $y = \frac{(6-x)}{2} = \frac{(6-(3))}{2} = \frac{3}{2}$

14. An open box is to be made from a rectangular piece of material by cutting equal squares from each corner and turning up the sides. Find the dimensions of the box of maximum volume if the material has dimensions of 2 feet by 3 feet.




15. Find the dimensions of the rectangle of maximum area that can be inscribed in a semicircle of radius a. [Hint : you may find it easier to maximize the *square of the area.* Of course, if the *square of the area* of the rectangle is a maximum, then the *area* of the rectangle is also a maximum.]

16. An indoor physical fitness room consists of a rectangular region with a semicircle on each end. If the perimeter of the room is to be a running track 200 meters in length, find the dimensions that will make the area of this rectangular region as large as possible.




17. A net enclosure for golf practice is open at one end, as shown in the figure below. Find the dimensions that require the least amount of netting if the volume of the enclosure is to be $\frac{250}{3}$ cubic meters, and the enclosure has a square cross section.