MTH 1126 - Test #2 - Solutions

SPRING 2019

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute the arclength of the graph of the function $f(x) = x^4 + \frac{1}{32}x^{-2}$ from the point $\left(1, \frac{33}{32}\right)$ to the point $\left(2, \frac{2049}{128}\right)$.

Arclength $= \int_a^b \sqrt{1 + (f'(x))^2}\,dx$

$f'(x) = 4x^3 - \frac{1}{16}x^{-3}$

$(f'(x))^2 = \left(4x^3 - \frac{1}{16}x^{-3}\right)^2 = 16x^6 - \frac{1}{2} + \frac{1}{256x^6}$

$$\text{Arclength} = \int_1^2 \sqrt{1 + (f'(x))^2}\,dx = \int_1^2 \sqrt{1 + \left(16x^6 - \frac{1}{2} + \frac{1}{256x^6}\right)}\,dx = \int_1^2 \sqrt{\left(16x^6 + \frac{1}{2} + \frac{1}{256x^6}\right)}\,dx$$

$$= \int_1^2 \sqrt{\left(\left(4x^3\right)^2 + \frac{1}{2} + \left(\frac{1}{16}x^{-3}\right)^2\right)}\,dx = \int_1^2 \sqrt{\left(4x^3 + \frac{1}{16}x^{-3}\right)^2}\,dx = \int_1^2 \left(4x^3 + \frac{1}{16}x^{-3}\right)dx$$

$$= \left[x^4 - \frac{1}{32}x^{-2}\right]_{x=1}^{x=2} = \left[x^4 - \frac{1}{32x^2}\right]_{x=1}^{x=2} = \left((2)^4 - \frac{1}{32(2)^2}\right) - \left((1)^4 - \frac{1}{32(1)^2}\right)$$

$$= \left(16 - \frac{1}{128}\right) - \left(1 - \frac{1}{32}\right) = 15 + \frac{3}{128} = \frac{1923}{128}$$

$$\boxed{\text{Arclength} = 15 + \frac{3}{128} = \frac{1923}{128}}$$

2. Find the area bounded by the graphs of $f(x) = x^2 - 4$ and $g(x) = x + 2$. (Partition the appropriate interval, sketch the i$^{\text{th}}$ rectangle, build the Riemann Sum, derive the appropriate integral.)

Graph the functions and find the points of intersection.

To find the points of intersection, set the y-coordinates equal to one another and solve for x.

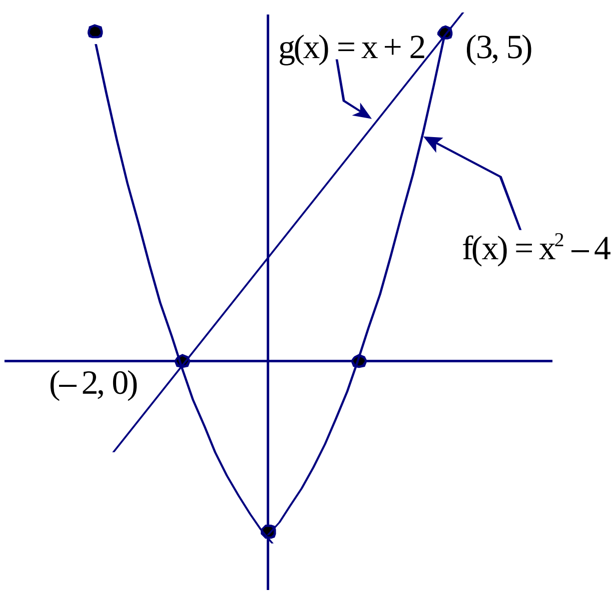
$y = x^2 - 4 = x + 2$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x+2)(x-3) = 0.$

$\Rightarrow x = -2$; and $x = 3$.

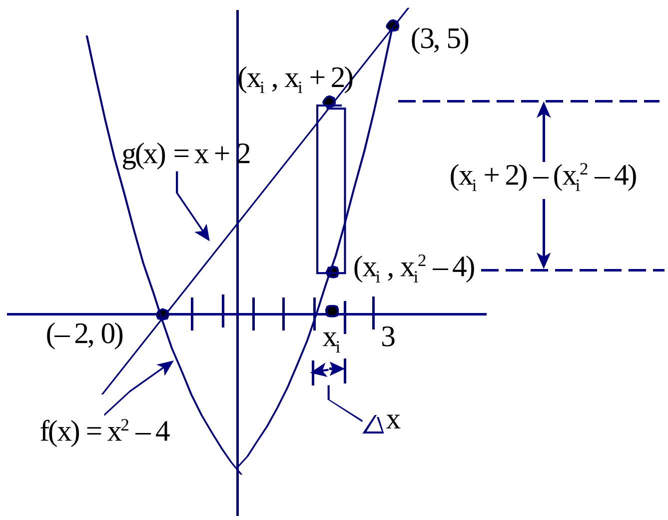
Points of intersection: $(-2, 0)$ and $(3, 5)$.



The rectangles span the interval $[-2, 3]$ on the x-axis, so we will partition that interval into sub-intervals of length Δx.

The area of the i^{th}. rectangle is $\underbrace{\left((x_i + 2) - \left(x_i^2 - 4\right)\right)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}} = \left(-x_i^2 + x_i + 6\right) \Delta x$

(see below)



To approximate the area of the bounded region, we add the areas of the rectangles:

$A \approx \sum_{i=1}^{n} \left(-x_i^2 + x_i + 6\right) \Delta x$

To get the exact area, we let $\Delta x \to 0$.

$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \left(-x_i^2 + x_i + 6\right) \Delta x = \int_{-2}^{3} \left(-x^2 + x + 6\right) dx = \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x\right]_{-2}^{3}$

$= \left(-\frac{1}{3}(3)^3 + \frac{1}{2}(3)^2 + 6(3)\right) - \left(-\frac{1}{3}(-2)^3 + \frac{1}{2}(-2)^2 + 6(-2)\right) = \frac{125}{6}$

i.e., bounded area $= \frac{125}{6}$

3. Use the "$f - g$" method to compute the area bounded by the graphs of $f(x) = \frac{1}{2}x$ and $g(x) = x^{\frac{1}{2}}$.

First, graph the functions and find the points of intersection.

$y = \frac{1}{2}x = x^{\frac{1}{2}}$

i.e., $\frac{1}{2}x = x^{\frac{1}{2}}$ (Square both sides)

$\Rightarrow \frac{1}{4}x^2 = x$

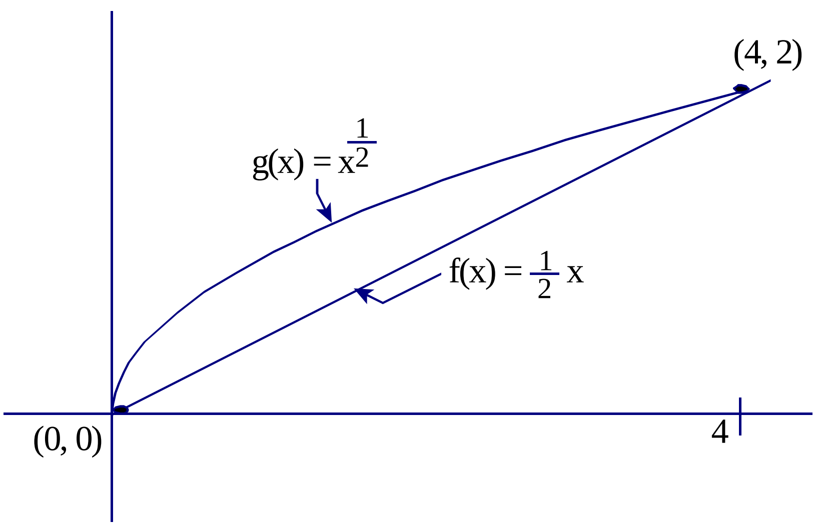
$\Rightarrow \frac{1}{4}x^2 - x = 0$

$\Rightarrow x^2 - 4x = 0$

$\Rightarrow x(x-4) = 0$

$x = 0$; $x = 4$

Points of intersection are $(0,0)$ and $(4,2)$.



The bounded region spans the interval $[0,4]$ on the x-axis. Over this interval, $g(x) = x^{\frac{1}{2}}$ is greater than $f(x) = \frac{1}{2}x$. Hence the area is given by:

$\int_0^4 \left(x^{\frac{1}{2}} - \frac{1}{2}x\right) dx = \left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} - \frac{1}{2}\frac{x^2}{2}\right]_0^4 = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^2\right]_0^4$

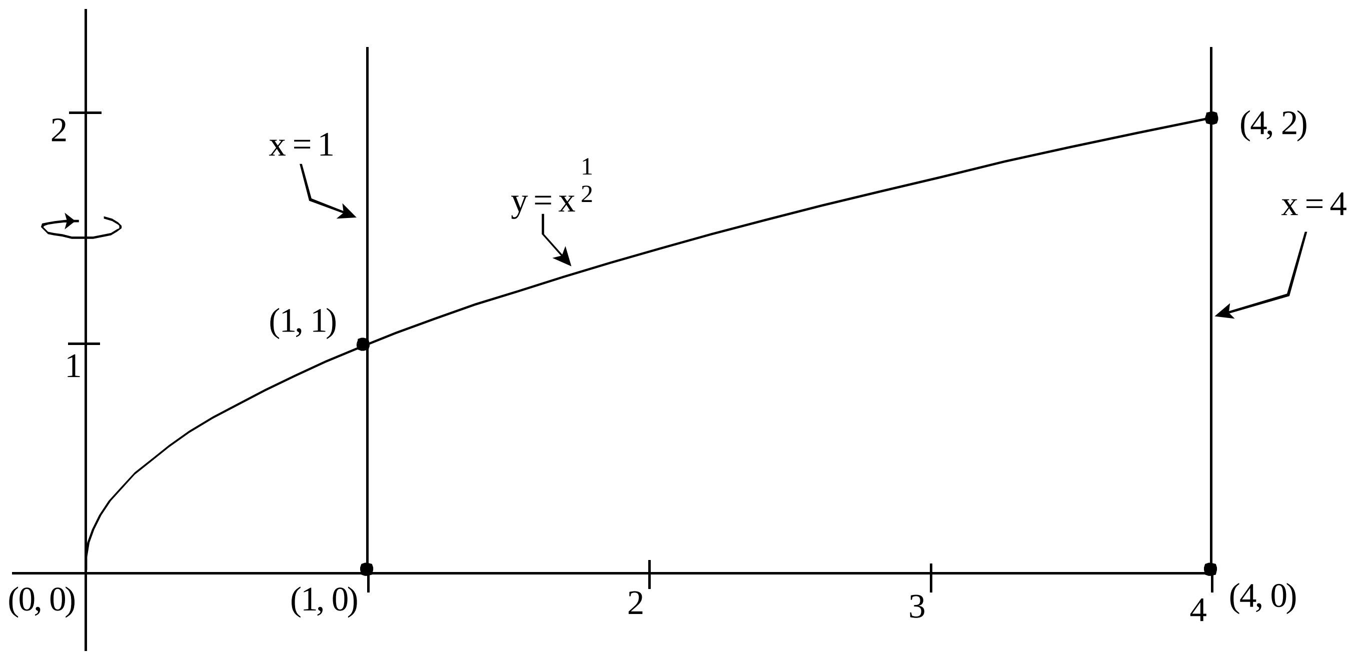
$= \left(\frac{2}{3}(4)^{\frac{3}{2}} - \frac{1}{4}(4)^2\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}} - \frac{1}{4}(0)^2\right) = \left(\frac{2}{3}(8) - 4\right) = \frac{4}{3}$

i.e., bounded area $= \frac{4}{3}$

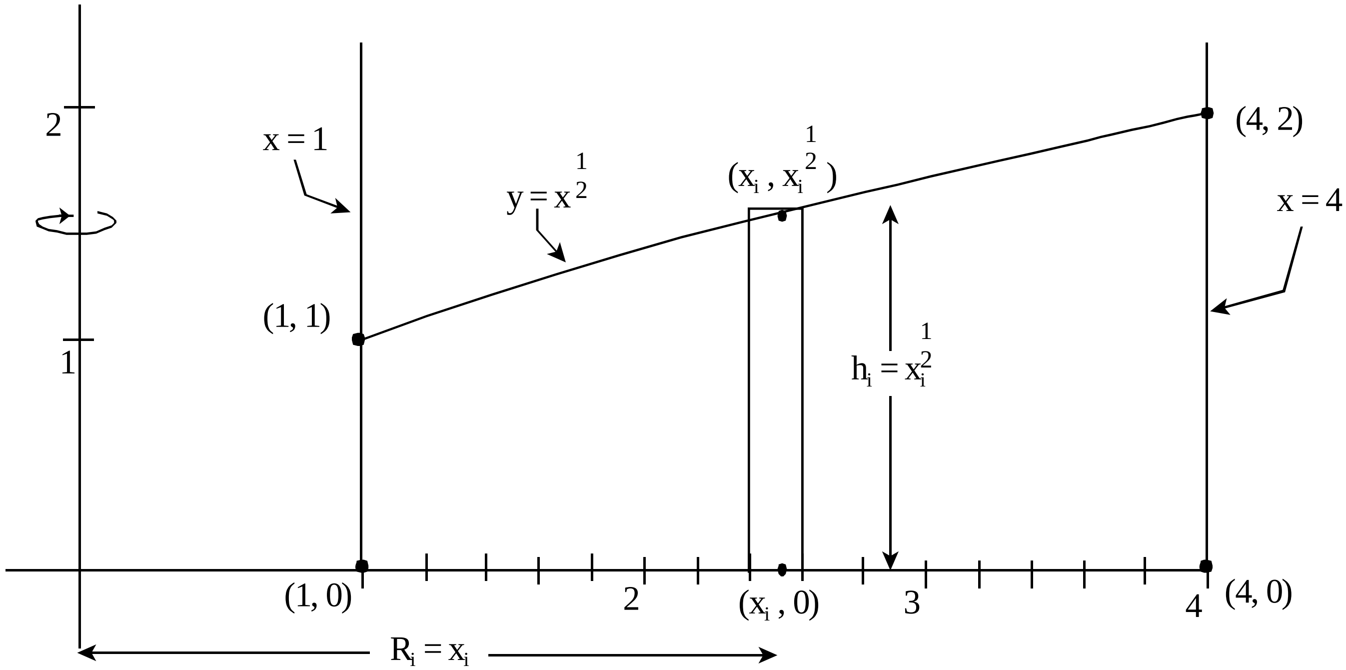
4. Use the "shell method" to compute the volume of the solid of revolution generated by revolving the region bounded by the graphs of $f(x) = x^{\frac{1}{2}}$, $x = 1$, $x = 4$, and the x-axis, about the y-axis. (For your information: the equation of the y-axis is $x = 0$.)

   Use the "five step method" (partition the interval, sketch the i^{th} rectangle, form the sum, take the limit)

   (a) 1. First, we'll graph the bounded region.



2. Next, we sketch a rectangle of width Δx parallel ("shell-parallel") to the axis of revolution, and we partition the interval spanned by the rectangles.



3. Revolve the i[th] rectangle about the axis of revolution and compute the volume of the i[th] shell, Vol_i

   $Vol_i = 2\pi R_i h_i \Delta x = 2\pi\,(x_i)\left(x_i^{\frac{1}{2}}\right)\Delta x = 2\pi x_i^{\frac{3}{2}}\Delta x$

4. Approximate the volume of the solid by adding up the volumes of the shells

   $Vol \approx \sum_{i=1}^{n} 2\pi x_i^{\frac{3}{2}}\Delta x$

5. Let $\Delta x \to 0$

   $Vol = \lim_{\Delta x\to 0}\sum_{i=1}^{n} 2\pi x_i^{\frac{3}{2}}\Delta x = \int_{x=1}^{x=4} 2\pi x^{\frac{3}{2}}dx = 2\pi\left[\frac{x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}\right]_{x=1}^{x=4} = \frac{4}{5}\pi\left[x^{\frac{5}{2}}\right]_{x=1}^{x=4}$

   $= \frac{4}{5}\pi\left[(4)^{\frac{5}{2}}\right] - \frac{4}{5}\pi\left[(1)^{\frac{5}{2}}\right] = \frac{4}{5}\pi\,[32] - \frac{4}{5}\pi\,[1] = \frac{124\pi}{5}$

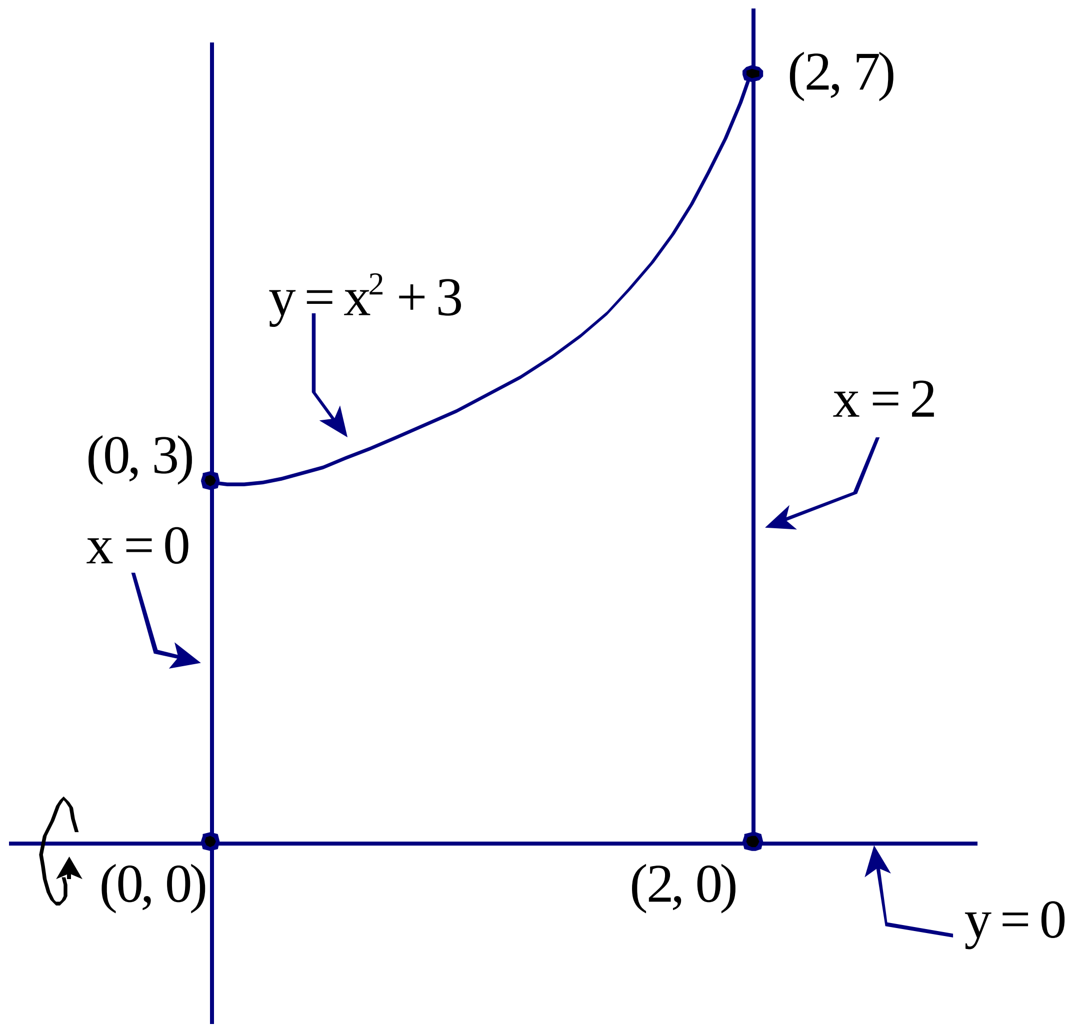
   $\boxed{Vol = \frac{124\pi}{5}}$

5. Use the "disc method" to compute the volume of the solid of revolution generated by revolving the region described below about the x-axis.

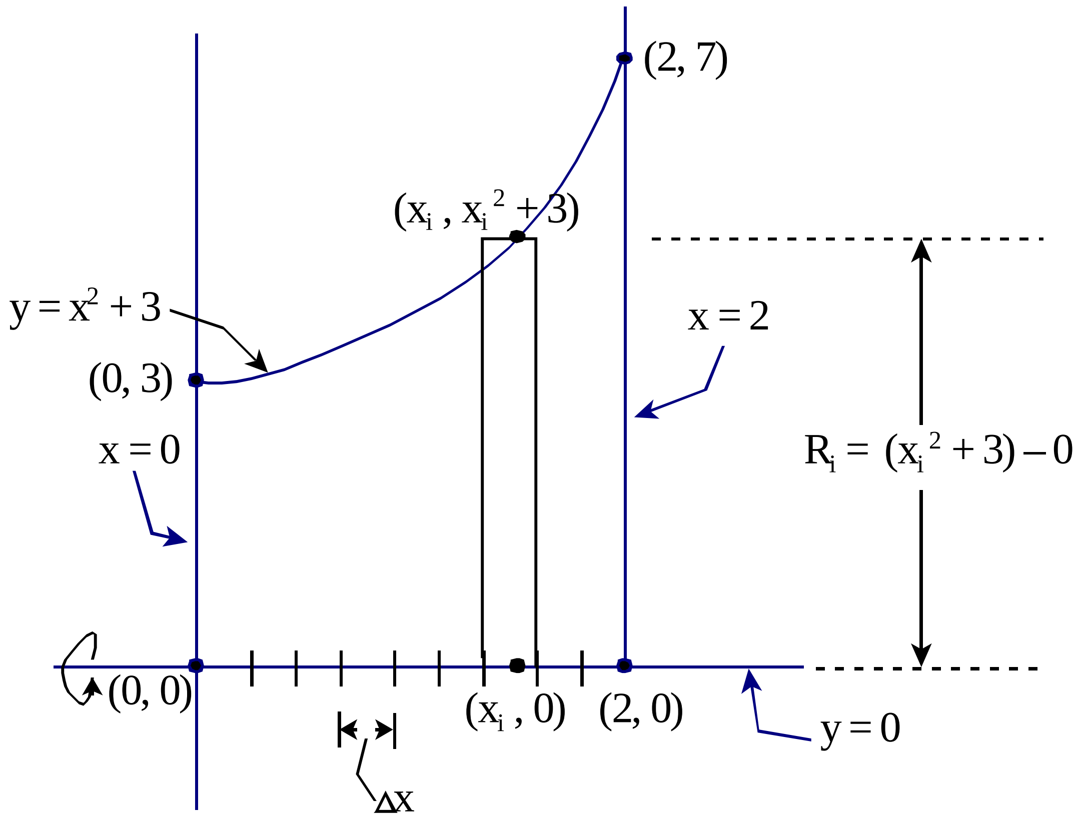
The region lies to the right of the y-axis and is bounded by the graph $f(x) = x^2 + 3$, the y-axis, the x-axis, and the line $x = 2$.

Use the "five step method" (partition the interval, sketch the i$^{\text{th}}$ rectangle, form the sum, take the limit)

(a) 1. First, we'll graph the bounded region.



2. Next, we sketch a rectangle of width Δx perpendicular (perpen-"disc"-ular) to the axis of revolution, and we partition the interval spanned by the rectangles.



3. Revolve the i$^{\text{th}}$ rectangle about the axis of revolution and compute the volume of the i$^{\text{th}}$ disc, Vol_i

   $Vol_i = \pi R_i^2 \Delta x = \pi \left(x_i^2 + 3\right)^2 \Delta x = \pi \left(x_i^4 + 6x_i^2 + 9\right) \Delta x$

4. Approximate the volume of the solid by adding up the volumes of the discs

   $Vol \approx \sum_{i=1}^{n} \pi \left(x_i^4 + 6x_i^2 + 9\right) \Delta x$

5. Let $\Delta x \to 0$

   $Vol = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi \left(x_i^4 + 6x_i^2 + 9\right) \Delta x = \int_{x=0}^{x=2} \pi \left(x^4 + 6x^2 + 9\right) dx$

   $= \pi \left[\frac{1}{5}x^5 + 2x^3 + 9x\right]_{x=0}^{x=2} = \pi \left[\frac{1}{5}(2)^5 + 2(2)^3 + 9(2)\right] - \pi \left[\frac{1}{5}(0)^5 + 2(0)^3 + 9(0)\right]$

   $= \pi \left[\frac{32}{5} + 16 + 18\right] - 0 = \frac{202\pi}{5}$

   $$\boxed{Vol = \frac{202\pi}{5}}$$