MTH 4441 Homework #4 - Solutions

DUE: MONDAY, SEPTEMBER 20, 2021

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Homework Exercises

1. Let $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, and let $(\mathbb{Z}_6, \oplus)$ be a group, where $\oplus$ is addition modulo 6. Construct the group table.

   Remark: The group $(\mathbb{Z}_6, \oplus)$ is called the **additive group of integers modulo 6.**

   Remark: In general, given $n \geq 2$, $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, and the group $(\mathbb{Z}_n, \oplus)$ is called the **additive group of integers modulo n.** ($\oplus$ is addition modulo n.)

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

2. Construct the group table for $(\mathbb{Z}_7, \oplus)$.

$\oplus$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

3. Let $U_5 = \{1, 2, 3, 4\}$, and let $(U_5, \odot)$ be a group, where $\odot$ is multiplication modulo 5. Construct the group table.

   Remark: The group $(U_5, \odot)$ is called the **multiplicative group of integers modulo 5.**

   Remark: In general, given $n \geq 2$, $U_n = \{1, \ldots, n-1\}$, and the group $(U_n, \odot)$ is called the **multiplicative group of integers modulo n.** ($\odot$ is multiplication modulo n.)

   In $(U_5, \odot)$, the operation $\odot$ is multiplication modulo 5

$\odot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

4. Construct the group table for $(U_3, \odot)$.

   In $(U_3, \odot)$, the operation $\odot$ is multiplication modulo 3

$\odot$	1	2
1	1	2
2	2	1

5. Construct the group table for $(U_7, \odot)$.

   In $(U_7, \odot)$, the operation $\odot$ is multiplication modulo 7

$\odot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

6. Construct the group table for $(U_6, \odot)$.

In $(U_6, \odot)$, the operation $\odot$ is multiplication modulo 6

$\odot$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

(a) Is $(U_6, \odot)$ actually a group? Why or why not?

$(U_6, \odot)$ is NOT a group

There are a few things that disqualify $(U_6, \odot)$ from being a group

1. U_6 is not closed under $\odot$. For example: $2 \odot 3 = 0$
2. None of the elements of U_6 appear at least once in every row and in every column. For example, 1 does not appear in the row headed by 2
3. The elements 2, 3, 4 appear more than once in some rows and columns. For example, 3 appears three times in the row and the column headed by 3.
4. The elements 2, 3, 4 do not have an inverse. This can be seen by the fact that the identity 1 does not appear in the rows and the columns that are headed by 2, 3, and 4.

(b) Does the equation $2x = 3$ have a solution in $(U_6, \odot)$? If not, what do you perceive the problem to be?

The equation $2x = 3$ does NOT have a solution in $(U_6, \odot)$. This is shown by the fact that the element 3 does not appear in the row headed by the element 2.

This is symptomatic of the fact that 2 does not have an inverse in U_6 and that U_6 is not closed under $\odot$.

If 2 DID have an inverse (let's denote it as 2^{-1}), then we could multiply both sides of the equation $2 \odot x = 3$ by 2^{-1}.

This would yield: $2^{-1} \odot (2 \odot x) = 2^{-1} \odot 3 \Rightarrow (2^{-1} \odot 2) \odot x = 2^{-1} \odot 3$

$\Rightarrow 1 \odot x = 2^{-1} \odot 3 \Rightarrow x = 2^{-1} \odot 3$.

Provided that U_6 is closed under $\odot$, $x = 2^{-1} \odot 3$ would be an element of U_6

Thus, $x = 2^{-1} \odot 3$ would be the solution to the equation$2 \odot x = 3$

To verify that $x = 2^{-1} \odot 3$ is the solution of the equation $2 \odot x = 3$, observe:

$2 \odot x = 2 \odot (2^{-1} \odot 3) = (2 \odot 2^{-1}) \odot 3 = 1 \odot 3 = 3$

i.e., $2 \odot x = 3$

7. Construct the group table for $(U_4, \odot)$.

In $(U_4, \odot)$, the operation $\odot$ is multiplication modulo 4

$\odot$	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

(a) Is $(U_4, \odot)$ actually a group? Why or why not?

$(U_4, \odot)$ is NOT a group

There are a few things that disqualify $(U_4, \odot)$ from being a group

1. U_4 is not closed under $\odot$. For example: $2 \odot 2 = 0$
2. The elements 1 and 3 of U_4 do NOT appear at least once in every row and in every column. The elements 1 and 3 do not appear in the row or the column headed by 2.
3. The elements 2 appears more than once in the row and column headed by 2.
4. The elements 2 does not have an inverse. This can be seen by the fact that the identity 1 does not appear in the row and the column that are headed by 2.

(b) Does the equation $2x = 3$ have a solution in $(U_4, \odot)$? If not, what do you perceive the problem to be?

The $2x = 3$ does NOT have a solution in $(U_4, \odot)$, as is seen from the fact that 3 does not appear in the row headed by 2.

Again, this is symptomatic of the fact that 2 does not have an inverse in U_4 and that U_4 is not closed under $\odot$.

(c) Under what conditions is $(U_n, \odot)$ a group? (Formulate a hypothesis.)

From the examples that we've seen, it would be reasonable to hypothesize that $(U_n, \odot)$ is a group exactly when n is prime.

8. Determine whether the table below defines a group for $G = \{a, b, c\}$. (State why or why not.)

$*$	a	b	c
a	a	b	c
b	b	a	c
c	c	b	a

Since the elements b and c each appear more than once in a column, the table does NOT define a group

Clearly, a is the identity. Therefore, there should not exist another element x such that $x * c = c$. And yet, $b * c = c$

Similarly, the element c is such that $c * b = b$

9. Determine whether the table below defines a group for $G = \{a, b, c\}$. (State why or why not.)

$*$	a	b	c
a	a	b	c
b	b	b	c
c	c	c	c

This table does NOT define $(G, *)$ as a group.

1. The elements b and c appear more than once in the rows and columns headed by b and c.

2. The identity 1 does not appear in any row or column headed by b or c, which indicates that neither b nor c has an inverse.