MTH 1126 - Practice Test #3 - Solutions

Fall 2015

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute $\int \frac{\ln(\sqrt{x})}{\sqrt{x}}\,dx$ using u-substitution:

$$\int \frac{\ln(\sqrt{x})}{\sqrt{x}}\,dx = \int \frac{\ln\left(x^{\frac{1}{2}}\right)}{x^{\frac{1}{2}}}\,dx = \int \underbrace{\ln\left(x^{\frac{1}{2}}\right)}_{\ln(u)}\underbrace{x^{-\frac{1}{2}}dx}_{2du} = \int \ln(u)\,2du = 2\int \ln(u)\,du$$

$$= 2\left[u\ln(u) - u\right] + C = 2\left(x^{\frac{1}{2}}\ln\left(x^{\frac{1}{2}}\right) - x^{\frac{1}{2}}\right) + C$$

$$\boxed{\begin{array}{lcl} u & = & x^{\frac{1}{2}} \\ \frac{du}{dx} & = & \frac{1}{2}x^{-\frac{1}{2}} \\ du & = & \frac{1}{2}x^{-\frac{1}{2}}dx \\ 2du & = & x^{-\frac{1}{2}}dx \end{array}}$$

$$\boxed{\text{i.e., } \int \frac{\ln(\sqrt{x})}{\sqrt{x}}\,dx = 2\left(x^{\frac{1}{2}}\ln\left(x^{\frac{1}{2}}\right) - x^{\frac{1}{2}}\right) + C}$$

2. Compute $\int \frac{\ln(\sqrt{x})}{\sqrt{x}}\,dx$ using Integration by Parts:

$$\int \frac{\ln(\sqrt{x})}{\sqrt{x}}\,dx = \int \frac{\ln\left(x^{\frac{1}{2}}\right)}{x^{\frac{1}{2}}}\,dx = \int \underbrace{\ln\left(x^{\frac{1}{2}}\right)}_{u}\underbrace{x^{-\frac{1}{2}}dx}_{dv} = \int u\,dv = uv - \int v\,du$$

$$= \underbrace{\ln\left(x^{\frac{1}{2}}\right)}_{u}\cdot\underbrace{2x^{\frac{1}{2}}}_{v} - \int \underbrace{2x^{\frac{1}{2}}}_{v}\cdot\underbrace{\frac{1}{2x}dx}_{du} = 2x^{\frac{1}{2}}\ln\left(x^{\frac{1}{2}}\right) - \int x^{-\frac{1}{2}}dx$$

$$= 2x^{\frac{1}{2}}\ln\left(x^{\frac{1}{2}}\right) - 2x^{\frac{1}{2}} + C$$

u	$=$	$\ln\left(x^{\frac{1}{2}}\right)$		dv	$=$	$x^{-\frac{1}{2}}dx$
$\frac{du}{dx}$	$=$	$\frac{1}{x^{\frac{1}{2}}}\cdot\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x}$		$\int dv$	$=$	$\int x^{-\frac{1}{2}}dx$
du	$=$	$\frac{1}{2x}dx$		v	$=$	$2x^{\frac{1}{2}}$

$$\boxed{\text{i.e., } \int \frac{\ln(\sqrt{x})}{\sqrt{x}}\,dx = 2\left(x^{\frac{1}{2}}\ln\left(x^{\frac{1}{2}}\right) - x^{\frac{1}{2}}\right) + C}$$

3. Compute: $\int x \ln(x)\, dx =$

Use Integration by Parts:

$$\int x \ln(x)\, dx = \int \underbrace{\ln(x)}_{u} \underbrace{x\, dx}_{dv} = \int u dv = uv - \int v du = \underbrace{\ln(x)}_{u} \cdot \underbrace{\frac{1}{2}x^2}_{v} - \int \underbrace{\frac{1}{2}x^2}_{v} \cdot \underbrace{\frac{1}{x}dx}_{du}$$

$$= \tfrac{1}{2}x^2 \ln(x) - \tfrac{1}{2}\int x dx = \tfrac{1}{2}x^2 \ln(x) - \tfrac{1}{2}\tfrac{x^2}{2} + C = \tfrac{1}{2}x^2 \ln(x) - \tfrac{1}{4}x^2 + C$$

u	$=$	$\ln(x)$	dv	$=$	xdx
$\frac{du}{dx}$	$=$	$\frac{1}{x}$	$\int dv$	$=$	$\int xdx$
du	$=$	$\frac{1}{x}dx$	v	$=$	$\frac{1}{2}x^2$

i.e. $\int x \ln(x)\, dx = \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C$

4. $\int \sin^3(x) \cos^4(x)\, dx =$

i. Pull out a factor of $\sin(x)$ to serve as the "future du."

$$= \int \sin^2(x) \cos^4(x) \underbrace{\sin(x)\, dx}_{\text{“future dú}}$$

Note: We intend to let $u = \cos(x)$. Consequently, $du = -\sin(x)$

Convert remaining sines into cosines by

ii. Re-writing sines in terms of $\sin^2(x)$

Done.

iii. Replace $\sin^2(x)$ with $1 - \cos^2(x)$

$$= \int \underbrace{\left(1 - \cos^2(x)\right) \cos^4(x)}_{(1-u^2)u^4} \underbrace{\sin(x)\, dx}_{-du} = \int (1 - u^2)\, u^4 (-du) = \int (u^6 - u^4)\, du$$

$$= \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C = \tfrac{1}{7}\cos^7(x) - \tfrac{1}{5}\cos^5(x) + C$$

i.e., $\int \sin^3(x) \cos^4(x)\, dx = \frac{1}{7}\cos^7(x) - \frac{1}{5}\cos^5(x) + C$

5. $\int xe^{2x}dx$

When we perform integration by parts, one criterion for choosing u and dv tells us that we should let dv be the most complicated part of the integrand that can be integrated, and let u be everything else.

$\Rightarrow dv = e^{2x}dx$ and $u = x$.

Another criterion tells us that u should be a function whose derivative is simpler than itself, and dv should be what's left over.

$\Rightarrow u = x$ and $dv = e^{2x}dx$.

Consequently, we let

u	$=$	x	dv	$=$	$e^{2x}dx$
$\frac{du}{dx}$	$=$	1	$\int dv$	$=$	$\int e^{2x}dx$
du	$=$	$1dx$	v	$=$	$\frac{1}{2}e^{2x}$

Thus we have:

$$\int \underbrace{x}_{u}\underbrace{e^{2x}dx}_{dv} = \int u\,dv = uv - \int v\,du = x\left(\tfrac{1}{2}e^{2x}\right) - \int\left(\tfrac{1}{2}e^{2x}\right)dx = \tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x}dx$$

$$= \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x}dx + C$$

$$\boxed{\text{i.e., } \int xe^{2x}\,dx = \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x}dx + C}$$

6. $\int x^2 e^{3x} dx =$

For reasons similar to those given in the previous problem, we let

u	$=$	x^2	dv	$=$	$e^{3x}dx$
$\frac{du}{dx}$	$=$	$2x$	$\int dv$	$=$	$\int e^{3x}dx$
du	$=$	$2x\,dx$	v	$=$	$\frac{1}{3}e^{3x}$

Thus,

$$\int \underbrace{x^2}_{u} \underbrace{e^{3x}dx}_{dv} = \int u\,dv = uv - \int v\,du = x^2 \tfrac{1}{3}e^{3x} - \int \left(\tfrac{1}{3}e^{3x}\right)(2x\,dx) = \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{3}\int xe^{3x}dx$$

At this point, we must repeat the procedure, letting

u	$=$	x	dv	$=$	$e^{3x}dx$
$\frac{du}{dx}$	$=$	1	$\int dv$	$=$	$\int e^{3x}dx$
du	$=$	dx	v	$=$	$\frac{1}{3}e^{3x}$

Continuing, we have:

$$\tfrac{1}{3}x^2e^{3x} - \tfrac{2}{3}\int \underbrace{x}_{u} \underbrace{e^{3x}dx}_{dv} = \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{3}\int u\,dv = \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{3}\left[uv - \int v\,du\right]$$

$$= \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{3}\left[x\tfrac{1}{3}e^{3x} - \int \tfrac{1}{3}e^{3x}dx\right] = \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{9}xe^{3x} + \tfrac{2}{27}e^{3x} + C$$

i.e., $\int x^2e^{3x}dx = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$

7. $\int e^x \sin(x)\, dx$

Here, both factors of the integrand have derivatives that are cyclic, so it won't matter which we choose to be u and which we choose to be dv. Also, since both factors of the integrand have derivatives that are cyclic, we expect that we will integrate by parts more than once, and eventually end up with the original integral on the right side of the equation. We will then solve for the integral algebraically. Let

u	$=$	e^x	dv	$=$	$\sin(x)\, dx$
$\frac{du}{dx}$	$=$	e^x	$\int dv$	$=$	$\int \sin(x)\, dx$
du	$=$	$e^x dx$	v	$=$	$-\cos(x)$

Thus, we have:

$$\int \underbrace{e^x}_{u} \underbrace{\sin(x)\, dx}_{dv} = \int u\, dv = uv - \int v\, du = e^x(-\cos(x)) - \int (-\cos(x))\, e^x dx$$

$$= -e^x \cos(x) + \int e^x \cos(x)\, dx$$

At this point, we repeat the procedure, being sure not to "switch" our choices of u and dv (i.e we continue to let u be e^x and we let dv be the trig function). This yields:

u	$=$	e^x	dv	$=$	$\cos(x)\, dx$
$\frac{du}{dx}$	$=$	e^x	$\int dv$	$=$	$\int \cos(x)\, dx$
du	$=$	$e^x dx$	v	$=$	$\sin(x)$

Continuing, we have:

$$-e^x \cos(x) + \int \underbrace{e^x}_{u} \underbrace{\cos(x)\, dx}_{dv} = -e^x \cos(x) + \int u\, dv = -e^x \cos(x) + uv - \int v\, du$$

$$= -e^x \cos(x) + e^x \sin(x) - \int \sin(x)\, e^x dx = -e^x \cos(x) + e^x \sin(x) - \underbrace{\int e^x \sin(x)\, dx}_{\text{what we started with}}$$

We sum up what we've established so far:

$$\int e^x \sin(x)\, dx = -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\, dx$$

We can solve for $\int e^x \sin(x)\, dx$ algebraically.

$$\Rightarrow 2\int e^x \sin(x)\, dx = -e^x \cos(x) + e^x \sin(x)$$

$$\Rightarrow \int e^x \sin(x)\, dx = -\frac{1}{2} e^x \cos(x) + \frac{1}{2} e^x \sin(x) + C$$

$$\boxed{\text{i.e., } \int e^x \sin(x)\, dx = -\frac{1}{2} e^x \cos(x) + \frac{1}{2} e^x \sin(x) + C}$$

8. $\int \cos^3(x)\sin^4(x)\,dx$

(cosine to an odd power) Pull out a factor of $\cos(x)$ to serve as the "future du", and convert the rest of the cosines to sines using the identity $\sin^2(x)+\cos^2(x)=1$.

$\int \cos^3(x)\sin^4(x)\,dx = \int \cos^2(x)\sin^4(x)\underbrace{\cos(x)\,dx}_{\text{future } du} = \int \left(1-\sin^2(x)\right)\sin^4(x)\cos(x)\,dx.$

Let $u=\sin(x)$

$\Rightarrow du = \cos(x)\,dx$

Continuing, we have: $\int \underbrace{\left(1-\sin^2(x)\right)}_{1-u^2}\underbrace{\sin^4(x)}_{u^4}\underbrace{\cos(x)\,dx}_{du} = \int \left(1-u^2\right)u^4du = \int \left(u^4-u^6\right)du$

$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + C = \frac{1}{5}\sin^5(x) - \frac{1}{7}\sin^7(x) + C$

$\boxed{\text{i.e., } \int \cos^3(x)\sin^4(x)\,dx = \frac{1}{5}\sin^5(x) - \frac{1}{7}\sin^7(x) + C}$

9. $\int \sin^3(x)\,dx$

(sine to an odd power) Pull out a factor of $\sin(x)$ to serve as the "future du", and convert the rest of the sines to cosines using the identity $\sin^2(x) = 1-\cos^2(x)$.

$\int \sin^3(x)\,dx = \int \sin^2(x)\underbrace{\sin(x)\,dx}_{\text{future } du} = \int \left(1-\cos^2(x)\right)\sin(x)\,dx.$

Let $u=\cos(x)$

$\Rightarrow du = -\sin(x)\,dx$

$\Rightarrow -du = \sin(x)\,dx$

Continuing, we have:

$\int \underbrace{\left(1-\cos^2(x)\right)}_{1-u^2}\underbrace{\sin(x)\,dx}_{-du} = \int \left(1-u^2\right)(-du) = \int \left(u^2-1\right)du$

$$= \frac{1}{3}u^3 - u + C = \frac{1}{3}\cos^3(x) - \cos(x) + C$$

$\boxed{\text{i.e., } \int \sin^3(x)\,dx = \frac{1}{3}\cos^3(x) - \cos(x) + C}$

10. $\int \sin^2(x)\cos^2(x)\,dx$

(sine and cosine raised to even powers) Reduce the powers of sine and cosine by using the double angle formulas:

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \quad \text{and} \quad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

Continuing:

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)\,dx &= \int \left(\tfrac{1-\cos(2x)}{2}\right)\left(\tfrac{1+\cos(2x)}{2}\right)dx = \tfrac{1}{4}\int \left(1-\cos^2(2x)\right)dx \\
&= \tfrac{1}{4}\int \left(1-\tfrac{1+\cos(4x)}{2}\right)dx = \tfrac{1}{4}\int \left(1-\tfrac{1}{2}-\tfrac{1}{2}\cos(4x)\right)dx \\
&= \tfrac{1}{4}\int \left(\tfrac{1}{2}-\tfrac{1}{2}\cos(4x)\right)dx = \tfrac{1}{8}\int \left(1-\cos(4x)\right)dx \\
&= \tfrac{1}{8}\left(x-\tfrac{1}{4}\sin(4x)\right)+C = \tfrac{1}{8}x-\tfrac{1}{32}\sin(4x)+C
\end{aligned}$$

$\boxed{\text{i.e., } \int \sin^2(x)\cos^2(x)\,dx = \tfrac{1}{8}x-\tfrac{1}{32}\sin(4x)+C}$

11. $\int \tan^3(x)\sec^3(x)\,dx$

(tangent raised to odd power) Pull out a factor of $\sec(x)\tan(x)$ to serve as the "future du", convert the remaining tangents to secants using the identity $\tan^2(x) = \sec^2(x)-1$.

Letting $u = \sec(x)$ (and consequently $du = \sec(x)\tan(x)\,dx$), we have:

$$\begin{aligned}
\int \tan^3(x)\sec^3(x)\,dx &= \int \tan^2(x)\sec^2(x)\underbrace{\sec(x)\tan(x)\,dx}_{\text{future } du} \\
&= \int \underbrace{\left(\sec^2(x)-1\right)}_{u^2-1}\underbrace{\sec^2(x)}_{u^2}\underbrace{\sec(x)\tan(x)\,dx}_{du} = \int \left(u^2-1\right)u^2du \\
&= \int \left(u^4-u^2\right)du = \tfrac{1}{5}u^5-\tfrac{1}{3}u^3+C = \tfrac{1}{5}\sec^5(x)-\tfrac{1}{3}\sec^3(x)+C
\end{aligned}$$

$\boxed{\text{i.e., } \int \tan^3(x)\sec^3(x)\,dx = \tfrac{1}{5}\sec^5(x)-\tfrac{1}{3}\sec^3(x)+C}$

12. $\int \tan^3(x)\sec^4(x)\,dx$

(secant raised to even power) Pull out a factor of $\sec^2(x)$ to serve as the "future du", convert the remaining secants to tangents using the identity $\tan^2(x)+1=\sec^2(x)$.

Letting $u=\tan(x)$ (and consequently $du=\sec^2(x)\,dx$), we have:

$$\int \tan^3(x)\sec^4(x)\,dx = \int \tan^3(x)\sec^2(x)\underbrace{\sec^2(x)dx}_{\text{future } du} = \int \underbrace{\tan^3(x)}_{u^3}\underbrace{\left(\tan^2(x)+1\right)}_{u^2+1}\underbrace{\sec^2(x)\,dx}_{du}$$

$$= \int u^3\left(u^2+1\right)du = \int \left(u^5+u^3\right)du = \tfrac{1}{6}u^6+\tfrac{1}{4}u^4+C$$

$$= \tfrac{1}{6}\tan^6(x)+\tfrac{1}{4}\tan^4(x)+C$$

i.e., $\int \tan^3(x)\sec^4(x)\,dx = \frac{1}{6}\tan^6(x)+\frac{1}{4}\tan^4(x)+C$

Remark 1 *Alternatively, we could have pulled out a factor of* $\sec(x)\tan(x)$ *for the "future du", letting* $u=\sec(x)$*. This yields:*

$$\int \tan^3(x)\sec^4(x)\,dx = \int \tan^2(x)\sec^3(x)\underbrace{\sec(x)\tan(x)\,dx}_{du}$$

$$= \int \underbrace{\left(\sec^2-1\right)}_{u^2-1}\underbrace{\sec^3(x)}_{u^3}\underbrace{\sec(x)\tan(x)\,dx}_{du} = \int \left(u^2-1\right)u^3du$$

$$= \int \left(u^5-u^3\right)du = \tfrac{1}{6}u^6-\tfrac{1}{4}u^4+C = \tfrac{1}{6}\sec^6(x)-\tfrac{1}{4}\sec^4(x)+C$$

i.e., $\int \tan^3(x)\sec^4(x)\,dx = \frac{1}{6}\sec^6(x)-\frac{1}{4}\sec^4(x)+C$

13. $\int \frac{1}{\sqrt{9+4x^2}} dx$

Use trig substitution to get rid of the radical. We want to replace $9+4x^2$ with something of the form $a^2 + a^2 \tan^2(\theta)$.

$a^2 = 9$

$\Rightarrow a = 3$

Also:

$4x^2 = a^2 \tan^2(\theta)$

$\Rightarrow 4x^2 = 9 \tan^2(\theta)$

$\Rightarrow 2x = 3 \tan(\theta)$

$\Rightarrow x = \frac{3}{2} \tan(\theta)$

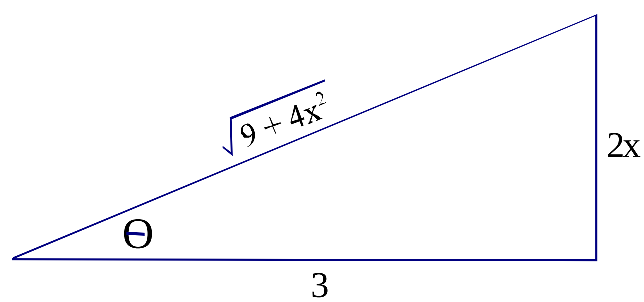
$\Rightarrow dx = \frac{3}{2} \sec^2(\theta)\, d\theta$

Thus, we have:

$$\int \frac{1}{\sqrt{9+4x^2}} dx = \int \frac{1}{\sqrt{9+9\tan^2(\theta)}} \frac{3}{2} \sec^2(\theta)\, d\theta = \int \frac{1}{\sqrt{9\sec^2(\theta)}} \frac{3}{2} \sec^2(\theta)\, d\theta = \int \frac{1}{3\sec(\theta)} \frac{3}{2} \sec^2(\theta)\, d\theta$$

$$= \frac{1}{2} \int \sec(\theta)\, d\theta = \frac{1}{2} \ln|\sec(\theta) + \tan(\theta)| + C$$

Observe: $x = \frac{3}{2}\tan(\theta) \Rightarrow \frac{2x}{3} = \tan(\theta) = \frac{\text{opp}}{\text{adj}}$. This yields the triangle below:



$$\Rightarrow \int \frac{1}{\sqrt{9+4x^2}} dx = \frac{1}{2} \ln|\sec(\theta) + \tan(\theta)| + C = \frac{1}{2} \ln\left|\frac{\sqrt{9+4x^2}}{3} + \frac{2x}{3}\right| + C$$

Simplifying, we have: $\frac{1}{2} \ln\left|\left(\frac{1}{3}\right)\left(\sqrt{9+4x^2} + 2x\right)\right| + C = \frac{1}{2}\left(\ln\left(\frac{1}{3}\right) + \ln\left|\sqrt{9+4x^2} + 2x\right|\right) + C$

$$= \underbrace{\frac{1}{2} \ln\left(\frac{1}{3}\right)}_{\text{Constant}} + \frac{1}{2} \ln\left|\sqrt{9+4x^2} + 2x\right| + C = \frac{1}{2} \ln\left|\sqrt{9+4x^2} + 2x\right| + C_1$$

$$\boxed{\text{i.e., } \int \frac{1}{\sqrt{9+4x^2}} dx = \frac{1}{2} \ln\left|\frac{\sqrt{9+4x^2}}{3} + \frac{2x}{3}\right| + C = \frac{1}{2} \ln\left|\sqrt{9+4x^2} + 2x\right| + C_1}$$

14. $\int \frac{\sqrt{x^2-9}}{x}dx$

Use trig substitution to get rid of the radical. We want to replace x^2-9 with something of the form $a^2\sec^2(\theta)-a^2$.

$a^2 = 9$

$\Rightarrow a = 3$

Also:

$x^2 = a^2\sec^2(\theta) \Rightarrow x^2 = 9\sec^2(\theta)$

$\Rightarrow x = 3\sec(\theta)$

$\Rightarrow dx = 3\sec(\theta)\tan(\theta)\,d\theta$

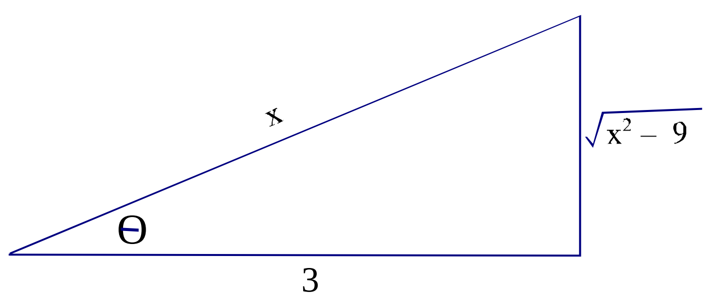
Thus, we have:

$$\int \frac{\sqrt{x^2-9}}{x}dx = \int \frac{\sqrt{9\sec^2(x)-9}}{3\sec(\theta)}3\sec(\theta)\tan(\theta)\,d\theta = \int \frac{\sqrt{9\tan^2(x)}}{3\sec(\theta)}3\sec(\theta)\tan(\theta)\,d\theta$$

$$= \int \frac{3\tan(\theta)}{3\sec(\theta)}3\sec(\theta)\tan(\theta)\,d\theta = 3\int \tan^2(\theta)\,d\theta = 3\int \left(\sec^2(\theta)-1\right)d\theta$$

$$= 3\tan(\theta) - 3\theta + C$$

Observe: $x = 3\sec(\theta) \Rightarrow \frac{x}{3} = \sec(\theta)$. This yields the triangle below:



Continuing where we left off:

$$\int \frac{\sqrt{x^2-9}}{x}dx = 3\tan(\theta) - 3\theta + C = 3\frac{\sqrt{x^2-9}}{3} - 3\sec^{-1}\left(\frac{x}{3}\right) + C$$

$$= \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right) + C$$

i.e., $\int \frac{\sqrt{x^2-9}}{x}dx = \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right) + C$

15. $\int \frac{11x+2}{2x^2-5x-3}dx$

Express as $\int \frac{11x+2}{(2x+1)(x-3)}dx$

Observe:

$\frac{11x+2}{(2x+1)(x-3)} = \frac{C_1}{2x+1} + \frac{C_2}{x-3}$

$\Rightarrow 11x + 2 = C_1\left(x-3\right) + C_2\left(2x+1\right)$

Solve for the constants, by plugging in "strategic values" of x.

$\boxed{x = 3}$ We have: $35 = 7C_2 \Rightarrow \mathbf{C}_2{=}5$

$\boxed{x = -\frac{1}{2}}$ We have: $-\frac{7}{2} = -\frac{7}{2}C_1 \Rightarrow \mathbf{C}_1{=}\,\mathbf{1}$.

Therefore:

$\int \frac{11x+2}{2x^2-5x-3}dx = \int \left(\frac{1}{2x+1} + \frac{5}{x-3}\right)dx = \int \frac{1}{2x+1}dx + \int \frac{5}{x-3}dx = \frac{1}{2}\ln|2x+1| + 5\ln|x-3| + C$

$\boxed{\text{i.e., } \int \frac{11x+2}{2x^2-5x-3}dx = \frac{1}{2}\ln|2x+1| + 5\ln|x-3| + C}$

16. $\int \frac{4x^2+x+1}{(x^2+1)(x-1)} dx =$

Let's decompose this using partial fractions.

$\frac{4x^2+x+1}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$

Remark 2 *Note that since x^2+1 is an* ***irreducible*** *quadratic, we put an $Ax+B$ above it.*

We multiply both sides by the least common denominator and get

$4x^2 + x + 1 = (Ax + B)(x - 1) + C(x^2 + 1)$

Solve for the constants, by plugging in "strategic values" of x.

$\boxed{x = 1}$ We have: $6 = 2C \Rightarrow \mathbf{C = 3}$

This gives us:

$4x^2 + x + 1 = (Ax + B)(x - 1) + 3(x^2 + 1)$

$\boxed{x = 0}$ We have: $1 = -B + 3 \Rightarrow B = 2$

This gives us:

$4x^2 + x + 1 = (Ax + 2)(x - 1) + 3(x^2 + 1)$

There don't seem to be any other really "strategic values" of x to plug in. Soooo . . . we will multiply everything out and compare coefficients of x^2 on both sides.

$4x^2 + x + 1 = (A + 3)x^2 + (2 - A)x + 3$

Comparing coefficients of x^2, we have: $A + 3 = 4$

$\Rightarrow A = 1$

Thus, we have:

$$\int \frac{4x^2+x+1}{(x^2+1)(x-1)} dx = \int \left(\frac{Ax+B}{x^2+1} + \frac{C}{x-1}\right) dx = \int \left(\frac{x+2}{x^2+1} + \frac{3}{x-1}\right) dx$$

$$= \int \frac{x+2}{x^2+1} dx + \int \frac{3}{x-1} dx = \int \frac{x}{x^2+1} dx + 2\int \frac{1}{x^2+1} dx + 3\int \frac{1}{x-1} dx$$

(Let $u = x^2 + 1$; then $du = 2x\,dx$ and $\frac{1}{2}du = x\,dx$.)

$= \int \frac{1}{u} \cdot \frac{1}{2} du + 2\int \frac{1}{x^2+1} dx + 3\int \frac{1}{x-1} dx = \frac{1}{2}\int \frac{1}{u} du + 2\int \frac{1}{x^2+1} dx + 3\int \frac{1}{x-1} dx =$

$\frac{1}{2}\ln|u| + 2\tan^{-1}(x) + 3\ln|x-1| + C = \frac{1}{2}\ln|x^2+1| + 2\tan^{-1}(x) + 3\ln|x-1| + C$

$\boxed{\text{i.e., } \int \frac{4x^2+x+1}{(x^2+1)(x-1)} dx = \frac{1}{2}\ln|x^2+1| + 2\tan^{-1}(x) + 3\ln|x-1| + C}$

17. $\lim_{x\to 0} \frac{\cos(x)+2x-1}{3x} \sim \frac{0}{0}$ So use L'Hôpital's Rule.

$\lim_{x\to 0} \frac{\cos(x)+2x-1}{3x} = \lim_{x\to 0} \frac{-\sin(x)+2}{3} = \frac{2}{3}$

$\boxed{\text{i.e., } \lim_{x\to 0} \frac{\cos(x)+2x-1}{3x} = \frac{2}{3}}$

18. $\lim_{x\to \frac{\pi}{2}^-} \frac{4\tan(x)}{1+\sec(x)} \sim \frac{\infty}{\infty}$ Use L'Hôpital's Rule.

$$\underbrace{\lim_{x\to \frac{\pi}{2}^-} \frac{4\tan(x)}{1+\sec(x)} = \lim_{x\to \frac{\pi}{2}^-} \frac{4\sec^2(x)}{\sec(x)\tan(x)}}_{\text{By L'Hôpital's Rule}} = \lim_{x\to \frac{\pi}{2}^-} \frac{4\sec(x)}{\tan(x)} = \lim_{x\to \frac{\pi}{2}^-} \frac{4}{\sin(x)} = 4$$

$\boxed{\text{i.e., } \lim_{x\to \frac{\pi}{2}^-} \frac{4\tan(x)}{1+\sec(x)} = 4}$

19. $\lim_{x\to 0} \frac{e^x+e^{-x}}{x^2} \sim \frac{2}{0}$ L'Hôpital's Rule does not apply!

$\lim_{x\to 0} \frac{e^x+e^{-x}}{x^2} = \frac{2}{\varepsilon} = \infty$ (where ε represents a minute positive value which is decreasing in value.)

$\boxed{\text{i.e., } \lim_{x\to 0} \frac{e^x+e^{-x}}{x^2} = \infty}$

20. $\lim_{x\to 0^+} x^2 \ln(x) \sim 0 \cdot (-\infty)$

We can rearrange this to fit the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and use L'Hôpital's Rule.

$\lim_{x\to 0^+} x^2 \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x^2}} \sim -\frac{\infty}{\infty}$

NOW use L'Hôpital's Rule.

$\lim_{x\to 0^+} x^2 \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0^+} \left(-\frac{x^2}{2}\right) = 0$

$\boxed{\text{i.e., } \lim_{x\to 0^+} x^2 \ln(x) = 0}$

21. $\lim_{x\to 0^+} (1+3x)^{\frac{1}{2x}} \sim 1^\infty$

Use the natural log to get $\frac{1}{2x}$ out of the exponent and either convert the expression into the form $\frac{0}{0}$, or the form $\frac{\infty}{\infty}$, so that we can use L'Hôpital's Rule.

Let $y = \lim_{x\to 0^+} (1+3x)^{\frac{1}{2x}}$ and note that if we find y, we will have found our limit.

$$y = \lim_{x\to 0^+} (1+3x)^{\frac{1}{2x}} \Rightarrow \ln(y) = \ln\left(\lim_{x\to 0^+} (1+3x)^{\frac{1}{2x}}\right) = \lim_{x\to 0^+} \ln\left[(1+3x)^{\frac{1}{2x}}\right]$$

$$= \lim_{x\to 0^+} \tfrac{1}{2x} \ln(1+3x) = \lim_{x\to 0^+} \tfrac{\ln(1+3x)}{2x} \sim \tfrac{0}{0}$$

NOW use L'Hôpital's Rule.

$$\ln(y) = \underbrace{\lim_{x\to 0^+} \frac{\ln(1+3x)}{2x} = \lim_{x\to 0^+} \frac{\frac{3}{1+3x}}{2}}_{\text{By L'Hôpital's Rule}} = \tfrac{3}{2}$$

What we've found is that $\ln(y) = \frac{3}{2}$. Using both sides as the exponent of e^x we get:

$$e^{\ln(y)} = e^{\frac{3}{2}} \Rightarrow y = e^{\frac{3}{2}}$$

i.e., $\lim_{x\to 0^+} (1+3x)^{\frac{1}{2x}} = e^{\frac{3}{2}}$

22. $\lim_{x\to 0^+} \left(\frac{1}{e^x-1} - \frac{1}{x}\right) \sim \infty - \infty$

L'Hôpital's Rule does not apply directly, but if we get a common denominator and combine the terms we might end up with something of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+} \left(\tfrac{1}{e^x-1} - \tfrac{1}{x}\right) = \lim_{x\to 0^+} \left(\tfrac{x}{x(e^x-1)} - \tfrac{e^x-1}{x(e^x-1)}\right) = \lim_{x\to 0^+} \left(\tfrac{x-e^x+1}{x(e^x-1)}\right) \sim \tfrac{0}{0}.$$

Use L'Hôpital's Rule!

$$\lim_{x\to 0^+} \left(\tfrac{1}{e^x-1} - \tfrac{1}{x}\right) = \underbrace{\lim_{x\to 0^+} \left(\frac{x-e^x+1}{x(e^x-1)}\right) = \lim_{x\to 0^+} \frac{1-e^x}{1\cdot(e^x-1)+e^x\cdot x}}_{\text{By L'Hôpital's (note the use of the product rule in the denominator)}} \sim \tfrac{0}{0}$$

Use L'Hôpital's again.

$$\lim_{x\to 0^+} \left(\tfrac{1}{e^x-1} - \tfrac{1}{x}\right) = \underbrace{\lim_{x\to 0^+} \frac{1-e^x}{1\cdot(e^x-1)+e^x\cdot x} = \lim_{x\to 0^+} \frac{-e^x}{e^x+e^x\cdot x+1\cdot e^x}}_{\text{By L'Hôpital's (note the use of the product rule in the denominator)}}$$

$$= \lim_{x\to 0^+} \tfrac{-e^x}{xe^x+2e^x} = -\tfrac{1}{2}$$

i.e., $\lim_{x\to 0^+} \left(\frac{1}{e^x-1} - \frac{1}{x}\right) = -\frac{1}{2}$