Integrals and Natural Logarithms #6 - Solutions

Fall 2013

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1. Compute: $\int \left(3x^6 + 5x^4 - 6x^2 + 2\sqrt{x}\right) dx =$

(Re-write) $\int \left(3x^6 + 5x^4 - 6x^2 + 2x^{\frac{1}{2}}\right) dx = 3\left[\frac{x^7}{7}\right] + 5\left[\frac{x^5}{5}\right] - 6\left[\frac{x^3}{3}\right] + 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$

$= \frac{3}{7}x^7 + x^5 - 2x^3 + \frac{4}{3}x^{\frac{3}{2}} + C$

i.e., $\int \left(3x^6 + 5x^4 - 6x^2 + 2\sqrt{x}\right) dx = \frac{3}{7}x^7 + x^5 - 2x^3 + \frac{4}{3}x^{\frac{3}{2}} + C$
Don't forget the "+C"

2. Compute: $\int (4\sec(x)\tan(x) - 8\csc(x)\cot(x))\, dx =$

$\int (4\sec(x)\tan(x) - 8\csc(x)\cot(x))\, dx = 4[\sec(x)] - 8[-\csc(x)] + C$

i.e., $\int (4\sec(x)\tan(x) - 8\csc(x)\cot(x))\, dx = 4\sec(x) + 8\csc(x) + C$
Don't forget the "+C"

3. Compute: $\int_{x=-1}^{x=2} \left(4x^3 - 8x^2 + 2\right) dx =$

$$\int_{x=-1}^{x=2} \underbrace{\left(4x^3 - 8x^2 + 2\right)}_{f(x)} dx = \underbrace{\left[4\left(\frac{x^4}{4}\right) - 8\left(\frac{x^3}{3}\right) + 2x\right]_{x=-1}^{x=2}}_{F(x)} = \underbrace{\left[x^4 - \frac{8}{3}x^3 + 2x\right]_{x=-1}^{x=2}}_{F(x)}$$

$$= \underbrace{\left[(2)^4 - \frac{8}{3}(2)^3 + 2(2)\right]}_{F(2)} - \underbrace{\left[(-1)^4 - \frac{8}{3}(-1)^3 + 2(-1)\right]}_{F(-1)} = 3$$

i.e., $\int_{x=-1}^{x=2} \left(4x^3 - 8x^2 + 2\right) dx = -3$

4. Compute: $\int (3\sin(x)+5)^5 \cos(x)\,dx$

   1. Is u-sub appropriate?

      a. Is there a composite function?

      Yes! $(3\sin(x)+5)^5$ (A function raised to a power is always a composite function!)

      Let $u =$ the "inner" of the composite function

      $\Rightarrow u = (3\sin(x)+5)$

      b. Is there an (approximate) function/derivative pair?

      Yes! $\underbrace{(3\sin(x)+5)}_{\text{function}} - - - - \rightarrow \underbrace{\cos(x)}_{\text{deriv}}$

      Let $u =$ the "function" of the function/deriv pair

      $\Rightarrow u = (3\sin(x)+5)$

      c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

      (i.e., do criteria **a** and **b** suggest the same choice of u ?)

      Yes! $\Rightarrow$ u-substitution is appropriate

   2. Compute du

$$\begin{array}{rcl} u & = & 3\sin(x)+5 \\ \Rightarrow \frac{du}{dx} & = & 3\cos(x) \\ \Rightarrow du & = & 3\cos(x)\,dx \\ \Rightarrow \frac{1}{3}du & = & \cos(x)\,dx \end{array}$$

   3. Analyze in terms of u and du

$$\int \underbrace{(3\sin(x)+5)^5}_{u^5}\underbrace{\cos(x)\,dx}_{\frac{1}{3}du} = \int u^5 \tfrac{1}{3}du = \tfrac{1}{3}\int u^5 du$$

   4. Integrate (in terms of u).

$$\tfrac{1}{3}\int u^5 du = \tfrac{1}{3}\left[\tfrac{u^6}{(6)}\right] + C = \tfrac{1}{18}u^6 + C$$

   5. Re-express in terms of the original variable, x.

$$\int (3\sin(x)+5)^5 \cos(x)\,dx = \underbrace{\frac{1}{18}(3\sin(x)+5)^6 + C}_{\frac{1}{18}u^6+C}$$

i.e., $\int (3\sin(x)+5)^5 \cos(x)\,dx = \frac{1}{18}(3\sin(x)+5)^6 + C$

5. Compute: $\int \cos(2x+1)\, dx =$

   1. Is u-sub appropriate?

      a. Is there a composite function?

      Yes! $\cos(2x+1)$

      $\nearrow$ $\nwarrow$

      outer inner

      Let $u =$ the "inner" of the composite function

      $\Rightarrow u = 2x+1$

      b. Is there an (approximate) function/derivative pair?

      ????????

      c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

      (i.e., do criteria **a** and **b** suggest the same choice of u ?)

      No – we don't see an (approximate) function/derivative pair.

      Nevertheless, we will proceed, based on the recommendation of Criterion a. (Note: Since Criterion b was not satisfied, u-substitution may not work.)

   2. Compute du

$$\begin{aligned} u &= 2x+1 \\ \Rightarrow \frac{du}{dx} &= 2 \\ \Rightarrow du &= 2\,dx \\ \Rightarrow \frac{1}{2}du &= dx \end{aligned}$$

   3. Analyze in terms of u and du

$$\int \underbrace{\cos(2x+1)}_{\cos(u)} \underbrace{dx}_{\frac{1}{2}du} = \int \cos(u)\ \frac{1}{2}du = \frac{1}{2}\int \cos(u)\, du$$

   4. Integrate (in terms of u).

$$\frac{1}{2}\int \cos(u)\, du = \frac{1}{2}\left[\sin(u)\right] + C = \frac{1}{2}\sin(u) + C$$

   5. Re-express in terms of the original variable, x.

$$\int \cos(2x+1)\ dx = \underbrace{\frac{1}{2}\sin(2x+1) + C}_{\frac{1}{2}\sin(u)+C}$$

i.e., $\int \cos(2x+1)\ dx = \frac{1}{2}\sin(2x+1) + C$

Remark: It turns out, that we really DID have an (approximate) function/derivative pair.

$\int \cos(2x+1)\ dx =$ is the same as $\int \cos(2x+1) \cdot 1\, dx$

Our (approximate) function/derivative pair is: $\underbrace{(2x+1)}_{\text{function}} - - - - \rightarrow \underbrace{1}_{\text{deriv}}$

6. Compute: $\int \frac{2x+1}{3x^2+3x}dx =$

$\int \frac{2x+1}{3x^2+3x}dx \underbrace{=}_{\text{re-write}} \int \frac{1}{3x^2+3x}\ (2x+1)\ dx$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

      Yes! $\frac{1}{3x^2+3x}$ is the same as $\left(3x^2+3x\right)^{-1}$, so it is a function raised to a power.

      Let $u =$ the "inner" of the composite function

      $\Rightarrow u = 3x^2 + 3x$

   b. Is there an (approximate) function/derivative pair?

      Yes! $\underbrace{(3x^2+3x)}_{\text{function}} - - - - - \rightarrow \underbrace{(2x+1)}_{\text{deriv}}$

      Let $u =$ the "function" of the function/deriv pair

      $\Rightarrow u = 3x^2 + 3x$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

      (i.e., do criteria **a** and **b** suggest the same choice of u ?)

      Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 3x^2 + 3x \\ \Rightarrow \frac{du}{dx} &= 6x + 3 \\ \Rightarrow du &= (6x+3)\,dx \\ \Rightarrow \frac{1}{3}du &= (2x+1)\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\underbrace{\int \frac{1}{3x^2+3x}}_{\frac{1}{u}} \underbrace{(2x+1)\ dx}_{\frac{1}{3}du} = \int \frac{1}{u}\frac{1}{3}du = \frac{1}{3}\int \frac{1}{u}du$$

4. Integrate (in terms of u).

   $\frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| + C$

5. Re-express in terms of the original variable, x.

$$\int \frac{2x+1}{3x^2+3x}dx = \underbrace{\frac{1}{3}\ln\left|3x^2+3x\right| + C}_{\frac{1}{3}\ln|u|+C}$$

$$\text{i.e., } \int \frac{2x+1}{3x^2+3x}dx = \frac{1}{3}\ln\left|3x^2+3x\right| + C$$

7. Compute: $\frac{d}{dx}\left[\ln\left(\csc(x)+4\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(\csc(x)+4\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{\csc(x)+4}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(-\csc(x)\cot(x)\right)}_{g'(x)} = -\frac{\csc(x)\cot(x)}{\csc(x)+4}$$

i.e., $\frac{d}{dx}\left[\ln\left(\csc(x)+4\right)\right] = -\frac{\csc(x)\cot(x)}{\csc(x)+4}$

8. Compute: $\frac{d}{dx}\left[\ln\left(5x^3-2x^2\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(5x^3-2x^2\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{5x^3-2x^2}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(15x^2-4x\right)}_{g'(x)} = \frac{15x^2-4x}{5x^3-2x^2} = \frac{15x-4}{5x^2-2x}$$

i.e., $\frac{d}{dx}\left[\ln\left(5x^3-2x^2\right)\right] = \frac{15x^2-4x}{5x^3-2x^2} = \frac{15x-4}{5x^2-2x}$

9. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2-1}{\cos(x)}}\right)\right] = \frac{d}{dx}\left[\ln\left(\left(\frac{x^2-1}{\cos(x)}\right)^{\frac{1}{2}}\right)\right]$

Remark: We can compute this derivative directly, in it's current form, but it would be much easier to use the *algebraic properties* of natural logarithms to simplify the expression first.

$$\frac{d}{dx}\left[\ln\left(\left(\frac{x^2-1}{\cos(x)}\right)^{\frac{1}{2}}\right)\right] = \underbrace{\frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{x^2-1}{\cos(x)}\right)\right]}_{\ln(a^n)=n\ln(a)} = \underbrace{\frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(x^2-1\right)-\ln\left(\cos(x)\right)\right)\right]}_{\ln\left(\frac{a}{b}\right)=\ln(a)-\ln(b)}$$

NOW we're ready to compute the derivative!

$$\frac{d}{dx}\left[\ln\left(\left(\frac{x^2-1}{\cos(x)}\right)^{\frac{1}{2}}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\left[\ln\left(x^2-1\right)-\ln\left(\cos(x)\right)\right]\right] = \frac{1}{2}\frac{d}{dx}\left[\left(\ln\left(x^2-1\right)-\ln\left(\cos(x)\right)\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{x^2-1}\cdot 2x - \frac{1}{\cos(x)}\left(-\sin(x)\right)\right] = \frac{x}{x^2-1} + \frac{1}{2}\frac{\sin(x)}{\cos(x)} = \frac{x}{x^2-1} + \frac{1}{2}\tan(x)$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2-1}{\cos(x)}}\right)\right] = \frac{x}{x^2-1} + \frac{1}{2}\frac{\sin(x)}{\cos(x)} = \frac{x}{x^2-1} + \frac{1}{2}\tan(x)$

10. Compute: $\int_{x=0}^{x=1} (3x-1)^4 \, dx$

1. Is u-sub appropriate?

    a. Is there a composite function?

    Yes! $(3x-1)^4$ (A function raised to a power is *always* a composite function!)

    Let $u =$ the "inner" of the composite function

    $\Rightarrow u = (3x-1)$

    b. Is there an (approximate) function/derivative pair?

    ?????

    c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

    (i.e., do criteria **a** and **b** suggest the same choice of u ?)

    No – we don't see an (approximate) function/derivative pair.

    Nevertheless, we will proceed, based on the recommendation of Criterion a. (Note: Since Criterion b was not satisfied, u-substitution may not work.)

2. Compute du

$$\begin{aligned} u &= 3x-1 \\ \Rightarrow \frac{du}{dx} &= 3 \\ \Rightarrow du &= 3\,dx \\ \Rightarrow \tfrac{1}{3}du &= dx \end{aligned}$$

When $x = 0$, $u = 3x - 1 = 3\,(0) - 1 = -1$
When $x = 1$, $u = 3x - 1 = 3\,(1) - 1 = 2$

3. Analyze in terms of u and du

$$\int_{x=0}^{x=1} \underbrace{(3x-1)^4}_{u^4} \underbrace{dx}_{\frac{1}{3}du} = \int_{u=-1}^{u=2} u^4 \cdot \tfrac{1}{3} du = \tfrac{1}{3} \int_{u=-1}^{u=2} u^4 du$$

Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$\tfrac{1}{3} \int_{u=-1}^{u=2} u^4 du = \tfrac{1}{3} \left[\tfrac{u^5}{5} \right]_{u=-1}^{u=2} = \tfrac{1}{15} \left[u^5 \right]_{u=-1}^{u=2} = \underbrace{\frac{1}{15}(2)^5}_{F(2)} - \underbrace{\frac{1}{15}(-1)^5}_{F(-1)} = \tfrac{1}{15}(32) - \tfrac{1}{15}(-1) = \tfrac{33}{15} = \tfrac{11}{5}$$

i.e., $\int_{x=0}^{x=1} (3x-1)^4 \, dx = \frac{33}{15} = \frac{11}{5}$

Remark: It turns out, that we really DID have an (approximate) function/derivative pair.

$\int_{x=0}^{x=1} (3x-1)^4 \, dx =$ is the same as $\int_{x=0}^{x=1} (3x-1)^4 \cdot 1\,dx$

Our (approximate) function/derivative pair is: $\underbrace{(3x-1)}_{\text{function}} - - - - \rightarrow \underbrace{1}_{\text{deriv}}$