MTH 3311 - Test #3 - Version #3 - Solutions

SPRING 2017

Pat Rossi Name ______________________________

Show CLEARLY how you arrive at your answers.

1. Find the general solution of the equation:
   $y'' + 3y' + 2y = 10e^{3x}$

   (a) First, find the solution to the complementary equation $y'' + 3y' + 2y = 0$

   $\Rightarrow m^2 + 3m + 2 = 0 \Rightarrow (m+2)(m+1) = 0 \Rightarrow m_1 = -2$ and $m_2 = -1$

   $\Rightarrow y_c = c_1 e^{-2x} + c_2 e^{-x}$

   (b) Find a Particular Solution

   For a particular solution, we imagine that $y_p = Ae^{3x}$

   $y_p' = 3Ae^{3x}$

   $y_p'' = 9Ae^{3x}$

   To find A, we plug these into the original equation, $y'' + 3y' + 2y = 10e^{3x}$.

   This yields:

   $$\underbrace{9Ae^{3x}}_{y''} + \underbrace{3Ae^{3x}}_{3y'} + \underbrace{2Ae^{3x}}_{2y} = 10e^{3x}$$

   $\Rightarrow (9A + 9A + 2A)\, e^{3x} = 10e^{3x}$

   i.e., $20A = 10$

   $\Rightarrow A = \frac{1}{2}$

   Hence, $y_p = \frac{1}{2}e^{3x}$

   (c) The general solution to the original equation is: $y = y_p + y_c$

   $$\boxed{\Rightarrow y = \frac{1}{2}e^{3x} + c_1 e^{-2x} + c_2 e^{-x}}$$

2. Find the general solution of the equation:
   $y'' - 2y' + y = -2\cos(3x) - 36\sin(3x)$

   (a) First, find the solution to the complementary equation $y'' - 2y' + y = 0$

   $\Rightarrow m^2 - 2m + 1 = 0 \Rightarrow (m-1)(m-1) = 0 \Rightarrow m_1 = 1$ and $m_2 = 1$

   i.e., $m = 1$ is a **Double Root!**

   To get two independent complementary solutions, we multiply the second term by x

   $\Rightarrow y_c = c_1 e^x + c_2 x e^x$

   (b) Find a Particular Solution

   For a particular solution, we imagine that $y_p = A\cos(3x) + B\sin(3x)$

   $y_p' = -3A\sin(3x) + 3B\cos(3x)$

   $y_p'' = -9A\cos(3x) - 9B\sin(3x)$

   To find A, we plug these into the original equation, $y'' - 2y' + y = -2\cos(3x) - 36\sin(3x)$

   This yields:

$$\underbrace{-9A\cos(3x) - 9B\sin(3x)}_{y''} - \underbrace{2(-3A\sin(3x) + 3B\cos(3x))}_{2y'} + \underbrace{A\cos(3x) + B\sin(3x)}_{y} = -2\cos(3x) - 36\sin(3x)$$

   $\Rightarrow (-9A - 6B + A)\cos(3x) + (-9B + 6A + B)\sin(3x) = -2\cos(3x) - 36\sin(3x)$

   $\Rightarrow (-9A - 6B + A) = -2$ and $(-9B + 6A + B) = -36$

   i.e., $(-8A - 6B) = -2$ (eq. 1) and $(-8B + 6A) = -36$ (eq.2)

   From eq.1, we have: $-6B = -2 + 8A \Rightarrow B = \frac{1}{3} - \frac{4}{3}A$

   Plugging this into eq. 2, we have:

   $\left(-8\left(\frac{1}{3} - \frac{4}{3}A\right) + 6A\right) = -36 \Rightarrow -\frac{8}{3} + \frac{32}{3}A + 6A = -36 \Rightarrow -8 + 32A + 18A = -108 \Rightarrow 50A = -100$

   $\Rightarrow A = -2$

   Plugging this into eq. 2, we have:

   $(-8B + 6(-2)) = -36 \Rightarrow -8B - 12 = -36 \Rightarrow -8B = -24$

   $\Rightarrow B = 3$

   Hence, $y_p = -2\cos(3x) + 3\sin(3x)$

   (c) The **general solution** is given by $y = y_p + y_c$

$$\boxed{\Rightarrow y = -2\cos(3x) + 3\sin(3x) + c_1 e^x + c_2 x e^x}$$

3. Find the general solution of the equation:

$$y'' - 12y' + 36y = \underbrace{e^{6x}\ln(x)}_{F(x)} \qquad \text{(Assume that } x > 0\text{)}$$

(a) First, find the solution to the complementary equation $y'' - 12y' + 36y = 0$

$\Rightarrow m^2 - 12m + 36 = 0 \Rightarrow (m-6)(m-6) = 0 \Rightarrow m_1 = 6$ and $m_2 = 6$

YIPES! Another **Double Root!**

To get two independent complementary solutions, we multiply the second term by x

$\Rightarrow y_c = c_1 e^{6x} + c_2 x e^{6x}$

(b) Find a Particular Solution

Since $F(x)$ (the right hand side of the equation) is NOT solely a linear combination of polynomials, $\sin(kx)$, $\cos(kx)$, or e^{kx}, we have to use **Variation of Parameters.**

We do this, by taking the complementary solution and replacing the constant coefficients with functions of x

$\Rightarrow y = A(x)e^{6x} + B(x)xe^{6x}$

We are allowed to impose two restrictions on the pair of functions $A(x)$ and $B(x)$.

The first restriction is that functions, $A(x)$ and $B(x)$, make the expression $y = A(x)e^{6x} + B(x)xe^{6x}$ a solution of the original differential equation.

$\Rightarrow y' = 6A(x)e^{6x} + A'(x)e^{6x} + B'(x)xe^{6x} + B(x)e^{6x} + 6B(x)xe^{6x}$

At this point, we impose the second of our restrictions, namely that $A'(x)e^{6x} + B'(x)xe^{6x} = 0$.

i.e. $A'(x)e^{6x} + B'(x)xe^{6x} = 0$ **(Restriction 2)**

Thus, we can rewrite y' as:

$y' = 6A(x)e^{6x} + B(x)e^{6x} + 6B(x)xe^{6x}$

$\Rightarrow y'' = 36A(x)e^{6x} + 6A'(x)e^{6x} + 6B(x)e^{6x} + B'(x)e^{6x} + 6B'(x)xe^{6x} + 6B(x)e^{6x} + 36B(x)xe^{6x}$

i.e., $y'' = 36A(x)e^{6x} + 6A'(x)e^{6x} + 12B(x)e^{6x} + B'(x)e^{6x} + 6B'(x)xe^{6x} + +36B(x)xe^{6x}$

Plug y, y', y'' into the original equation $y'' - 12y' + 36y = e^{6x}\ln(x)$

$$\begin{array}{rclllllll}
y'' & = & 36A(x)e^{6x} & +6A'(x)e^{6x} & +12B(x)e^{6x} & +B'(x)e^{6x} & +6B'(x)xe^{6x} & +36B(x)xe^{6x} \\
-12y' & = & -72A(x)e^{6x} & & -12B(x)e^{6x} & & & -72B(x)xe^{6x} \\
+36y & = & 36A(x)e^{6x} & & & & & +36B(x)xe^{6x} \\
\hline
e^{6x}\ln(x) & = & & +6A'(x)e^{6x} & & +B'(x)e^{6x} & +6B'(x)xe^{6x} & \text{(Eq.1)}
\end{array}$$

Subtracting $6 \cdot$ (Restriction 2) from Eq. 1, we have:

$$\begin{array}{rclll}
e^{6x}\ln(x) & = & 6A'(x)e^{6x} & +B'(x)e^{6x} & +6B'(x)xe^{6x} \\
0 & = & -6A'(x)e^{6x} & & -B'(x)xe^{6x} \\
\hline
e^{6x}\ln(x) & = & & B'(x)e^{6x} &
\end{array}$$

i.e., $B'(x)\, e^{6x} = e^{6x} \ln(x)$

$\Rightarrow B'(x) = \ln(x) \qquad \text{(Eq.2)}$

$\Rightarrow B(x) = \int B'(x)\, dx = \int \ln(x)\, dx = x \ln(x) - x + C_3$

i.e., $B(x) = x \ln(x) - x + C_3$

In order to find $A(x)$, our easiest option might be to plug $B'(x) = \ln(x)$ (from Eq.2) into Restriction 2, $A'(x)\, e^{6x} + B'(x)\, xe^{6x} = 0$

$\Rightarrow A'(x)\, e^{6x} + \ln(x)\, xe^{6x} = 0$

Notice that e^{6x} is a factor of both terms, and that e^{6x} is never equal to zero. Thus, we can divide both sides by e^{6x}.

$\Rightarrow A'(x) + x \ln(x) = 0$

$\Rightarrow A'(x) = -x \ln(x)$

$\Rightarrow A(x) = \int A'(x)\, dx = -\int x \ln(x)\, dx = -\frac{1}{2}x^2 \ln(x) + \frac{1}{4}x^2 + C_4$

i.e., $A(x) = \frac{1}{4}x^2 - \frac{1}{2}x^2 \ln(x) + C_4$

Thus, $y = A(x)\, e^{6x} + B(x)\, xe^{6x} = \left(\frac{1}{4}x^2 - \frac{1}{2}x^2 \ln(x) + C_4\right) e^{6x} + (x \ln(x) - x + C_3)\, e^{6x}$

$$\boxed{y = A(x)\, e^{6x} + B(x)\, xe^{6x} = \left(\frac{1}{4}x^2 - \frac{1}{2}x^2 \ln(x) + C_4\right) e^{6x} + (x \ln(x) - x + C_3)\, xe^{6x}}$$

Remark: $-\int x \ln(x)\, dx$ was computed using integration by parts:

$u = \ln(x)$	$dv = xdx$
$du = \frac{1}{x}dx$	$v = \frac{1}{2}x^2$

$$-\int x \ln(x)\, dx = -\left[\underbrace{\ln(x)\, \frac{1}{2}x^2}_{uv} - \underbrace{\int \frac{1}{2}x^2 \frac{1}{x} dx}_{\int vdu}\right] = -\left[\frac{1}{2}x^2 \ln(x) - \frac{1}{2}\int xdx\right] = \left[-\frac{1}{2}x^2 \ln(x) + \frac{1}{4}x^2 + C_4\right]$$