MTH 1126 - Test #4 - Solutions

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Show CLEARLY how you arrive at your answers.

In Exercises 1-2, Determine convergence/divergence. If the integral converges, find its value.

1. $\int_5^\infty \frac{1}{(x-1)^{\frac{1}{2}}}dx =$

$$\int_5^\infty \frac{1}{(x-1)^{\frac{1}{2}}}dx = \lim_{b\to\infty} \int_5^b \frac{1}{(x-1)^{\frac{1}{2}}}dx = \lim_{b\to\infty} \int_5^b (x-1)^{-\frac{1}{2}}\,dx = \lim_{b\to\infty} \left[\frac{(x-1)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right]_5^b$$

$$= \lim_{b\to\infty} \left[2(x-1)^{\frac{1}{2}}\right]_5^b = \lim_{b\to\infty} \left[2(b-1)^{\frac{1}{2}} - 2(5-1)^{\frac{1}{2}}\right]$$

$$= \lim_{b\to\infty} \left[2(b-1)^{\frac{1}{2}} - 2(4)^{\frac{1}{2}}\right] = \lim_{b\to\infty} \left[2(b-1)^{\frac{1}{2}} - 4\right]$$

$$= \infty - 4 = \infty$$

i.e., $\int_5^\infty \frac{1}{(x-1)^{\frac{1}{2}}}dx = \infty$ (Integral **Diverges**)

2. $\int_2^6 \frac{1}{(x-2)^{\frac{3}{2}}}dx =$

Because $\frac{1}{(x-2)^{\frac{3}{2}}}$ is discontinuous at $x = 2$, this is an improper integral.

$$\int_2^6 \frac{1}{(x-2)^{\frac{3}{2}}}dx = \lim_{a\to 2^+} \int_a^6 (x-2)^{-\frac{3}{2}}\,dx = \lim_{a\to 2^+} \int_a^6 \frac{(x-2)^{-\frac{1}{2}}}{-\frac{1}{2}}dx = \lim_{a\to 2^+} \left[-2(x-2)^{-\frac{1}{2}}\right]_a^6$$

$$= \lim_{a\to 2^+} \left[-\frac{2}{(x-2)^{\frac{1}{2}}}\right]_a^6 = \lim_{a\to 2^+} \left[-\frac{2}{(6-2)^{\frac{1}{2}}} - \left(-\frac{2}{(a-2)^{\frac{1}{2}}}\right)\right]$$

$$= \left[-1 + \frac{2}{(+\varepsilon)^{\frac{1}{2}}}\right] = +\infty$$

i.e. $\int_2^6 \frac{1}{(x-2)^{\frac{3}{2}}}dx = +\infty$ (Integral **Diverges**)

3. Determine convergence/divergence of the sequence whose n^{th} term is given by:

   $a_n = (-1)^{n+1}\frac{1}{n}$. (i.e., Determine convergence/divergence of the sequence

   $\left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}\ldots\right\}$.)

   Observe: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n+1}\frac{1}{n} = 0$

   Since $\lim_{n\to\infty} a_n$ is a finite real number, the sequence converges to that limit.

   $\left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=1}^{\infty}$ **converges** to 0.

4. Determine convergence/divergence of the given series. (Justify your answer!) **If the series converges, determine its sum.**

   $\sum_{n=1}^{\infty} \frac{2}{n^2+2n} =$

   "If the series converges, determine its sum." In general, there are only two types of convergent series whose sums we can compute: Geometric and "Telescoping (Collapsing) Sum."

   The series $\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$ is definitely NOT Geometric.

   Maybe it can be written as a "Telescoping (Collapsing) Sum."

   So let's see if we can express $a_n = \frac{2}{n^2+2n}$ as the difference of two terms.

   $\frac{2}{n^2+2n} = \frac{2}{n(n+2)} = \frac{C_1}{n} + \frac{C_2}{n+2}$

   i.e., $\frac{2}{n(n+2)} = \frac{C_1}{n} + \frac{C_2}{n+2}$

   $\Rightarrow \frac{2}{n(n+2)} n(n+2) = \frac{C_1}{n} n(n+2) + \frac{C_2}{n+2} n(n+2)$

   $\Rightarrow 2 = C_1(n+2) + C_2 n$

   $n = 0$ $\Rightarrow 2 = C_1(2)$

   $\Rightarrow C_1 = 1$

   $n = -2$ $\Rightarrow 2 = C_2(-2)$

   $\Rightarrow C_2 = -1$

Thus, $\frac{2}{n^2+2n} = \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$

$$\Rightarrow \sum_{n=1}^{N} \frac{2}{n^2+2n} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+2}\right) = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots$$

$$+ \left(\frac{1}{N-2} - \frac{1}{N}\right) + \left(\frac{1}{N-1} - \frac{1}{N+1}\right) + \left(\frac{1}{N} - \frac{1}{N+2}\right)$$

$$= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+2}\right) = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$$

i.e., $\sum_{n=1}^{\infty} \frac{2}{n^2+2n} = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+2}\right) = \lim_{N\to\infty} \left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right) = \frac{3}{2}$

i.e., $\sum_{n=1}^{\infty} \frac{2}{n^2+2n} = \frac{3}{2}$

In Exercises 5-6, determine convergence/divergence of the given series. (Justify your answers!) **If the series converges, determine its sum.**

5. $1 + \frac{4}{5} + \frac{16}{25} + \frac{64}{125} + \dots + \left(\frac{4}{5}\right)^n + \dots$

   "If the series converges, determine its sum." In general, there are only two types of convergent series whose sums we can compute: Geometric and "Telescoping Sum."

   Notice that each term after the first term is equal to $\frac{4}{5}$ times its predecessor.

   The series is geometric with ratio $r = \frac{4}{5}$

   Since $|r| < 1$, the series converges to $\frac{1^{\text{st}} \text{ term}}{1-r} = \frac{1}{1-\frac{4}{5}} = \frac{1}{\left(\frac{1}{5}\right)} = 5$

   The series **converges** to 5

6. $\sum_{n=1}^{\infty} \frac{n}{2n-1} =$

First, note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$

Since $\lim_{n\to\infty} a_n \neq 0$, the series **diverges.**

i.e., $\sum_{n=1}^{\infty} \frac{n}{n+5}$ **diverges** by the "n^{th} term Test."

In Exercises 7-9, determine convergence/divergence of the given series. (Justify your answers!)

7. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}+1}$

There are a few different ways that we can try to do this.

We can compare $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}+1}$ to $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$, which is a p-series with $p = \frac{2}{3} < 1$.

Hence $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$ diverges.

Since $\underbrace{\frac{1}{n^{\frac{2}{3}}+1}}_{a_n} < \underbrace{\frac{1}{n^{\frac{2}{3}}}}_{b_n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$ diverges, we can conclude nothing from the **Direct Comparison Test.**

(i.e. the fact that the "larger series" diverges in no way implies that the "smaller series" diverges.)

However, this does not preclude us from using the **Limit** Comarison Test.

Observe: $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\left(\frac{1}{n^{\frac{2}{3}}+1}\right)}{\left(\frac{1}{n^{\frac{2}{3}}}\right)}\right| = \lim_{n\to\infty} \frac{n^{\frac{2}{3}}}{n^{\frac{2}{3}}+1} = \lim_{n\to\infty} \frac{n^{\frac{2}{3}}}{n^{\frac{2}{3}}} = 1$

Since $0 < \lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| < \infty$, Both series "do the same thing."

Since $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{2}{3}}}$, is a divergent p-series (with $p = \frac{2}{3}$), $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{2}{3}}+1}$ diverges also, by the **Limit Comparison Test.**

i.e., $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{2}{3}}+1}$ **diverges** by the **Limit Comparison Test** with $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{2}{3}}}$

8. $\sum_{n=2}^{\infty} \frac{1}{n-1}$

There are a few ways to do this.

First, we can compare $\sum_{n=2}^{\infty} \frac{1}{n-1}$ with $\sum_{n=1}^{\infty} \frac{1}{n}$, which is the divergent Harmonic Series.

Since $\underbrace{\dfrac{1}{n-1}}_{a_n} > \underbrace{\dfrac{1}{n}}_{b_n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=2}^{\infty} \frac{1}{n-1}$ **diverges** by the Direct Comparison Test.

(i.e., since the "smaller series" diverges, the "larger series" diverges also.)

Alternatively: Applying the Limit Comparison Test, we have:

$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\left(\frac{1}{n-1}\right)}{\left(\frac{1}{n}\right)}\right| = \lim_{n\to\infty} \frac{n}{n-1} = 1$

Since $0 < \lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| < \infty$, Both series "do the same thing."

Since $\sum_{n=1}^{\infty} \frac{1}{n}$, is the divergent Harmonic Series, $\sum_{n=2}^{\infty} \frac{1}{n-1}$ diverges also, by the **Limit Comparison Test.**

Alternatively: $\int_2^{\infty} \frac{1}{n-1} dn = \lim_{b\to\infty} \int_2^b \underbrace{\dfrac{1}{n-1}}_{\frac{1}{u}} \underbrace{dn}_{du} = \lim_{b\to\infty} \left[\ln(n-1)\right]_2^b$ =

$\lim_{b\to\infty} \left[\ln(b-n) - \ln(2-1)\right] = \infty$

$\sum_{n=2}^{\infty} \frac{1}{n-1}$ **diverges** by the **Integral Test**

Alternatively: $\sum_{n=2}^{\infty} \frac{1}{n-1} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is the Harmonic Series, which **Diverges.**

$\sum_{n=2}^{\infty} \frac{1}{n-1}$ **diverges** by the **Direct Comparison** and **Limit Comparison** with $\sum_{n=2}^{\infty} \frac{1}{n}$
Or $\sum_{n=2}^{\infty} \frac{1}{n-1}$ **diverges** by the **Integral Test, or** because it's the **Harmonic Series**

9. Determine convergence/divergence of the given series. (Justify your answer!)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \tfrac{1}{2n} = \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \ldots$$

Observe: [1] $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n} = 0$

Also: $\frac{1}{2n} > \frac{1}{2(n+1)}$ i.e. $a_n > a_{n+1}$

Finally: the series is alternating.

By the Alternating Series Test, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n}$ **converges**

For exercises 10-11, choose one. (You can do the other for extra credit. (10 points))

10. Determine convergence/divergence of the given series. (Justify your answer!)

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{2n+1}}\right)^n$$

The n^{th} term, a_n is something **raised to the n^{th} power,** so this series is a good candidate for the n^{th} **Root Test.**

Observe: $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{\sqrt{2n+1}}\right)^n} = \lim_{n\to\infty}\left(\frac{1}{\sqrt{2n+1}}\right) = 0$

Since $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$, the series **converges.** by the n^{th} **Root Test.**

$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{2n+1}}\right)^n$ **converges** by the n^{th} **Root Test.**

11. Determine convergence/divergence of the given series. (Justify your answer!)

$$\sum_{n=1}^{\infty}\frac{5^{2n}}{n!}$$

The n^{th} term a_n contains a **factorial,** so this is a good candidate for the **Ratio Test.**

Observe: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{5^{2(n+1)}}{(n+1)!}}{\left(\frac{5^{2n}}{n!}\right)}\right| = \lim_{n\to\infty}\frac{5^{2(n+1)}}{(n+1)!}\frac{n!}{5^{2n}} = \lim_{n\to\infty}\frac{5^{2n+2}}{(n+1)!}\frac{n!}{5^{2n}} =$
$\lim_{n\to\infty}\frac{5^2}{n+1} = 0$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, the series **converges.**

$\sum_{n=1}^{\infty}\frac{5^{2n}}{n!}$ **converges** by the **Ratio Test.**