Proofs Involving Functions

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Instructions. Prove or disprove:

1. $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = 4x - 7$, is one to one.

   Proof. Suppose that $f(x_1) = f(x_2)$

   $\Rightarrow 4x_1 - 7 = 4x_2 - 7$

   $\Rightarrow 4x_1 = 4x_2$

   $\Rightarrow x_1 = x_2$

   i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

   Hence, $f(x)$ is one to one. ■

2. $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = 4x - 7$, is onto.

   Proof. We must show that given $y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ such that $f(x) = y$

   So let $y \in \mathbf{R}$ be given.

   Let $x = \underline{\frac{y+7}{4}}$

   Observe: $f(x) = 4x - 7 = \underline{4\left(\frac{y+7}{4}\right) - 7} = \underline{(y+7) - 7} = y$

   i.e., Given $y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ (namely $x = \underline{\frac{y+7}{4}}$) such that $f(x) = y$.

   Hence, $f(x)$ is one to one. ■

   Scratch Work We want x such that $f(x) = y$

   $\Rightarrow 4x - 7 = y$

   $\Rightarrow 4x = y + 7$

   $\Rightarrow x = \frac{y+7}{4}$

3. $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = 2x^2 + 4$, is one to one.

   This is FALSE.

   Observe: Given $x_1 = -1$ and $x_2 = 1$, we have $f(x_1) = 6 = f(x_2)$

   i.e., two distinct x-values get mapped to the same y-value.

   Hence, f is NOT one to one.

4. $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = 2x^2 + 4$, is onto.

   This is FALSE.

   Observe that $f(x) = 2x^2 + 4 \geq +4$ for all real numbers, x.

   Hence, given $y = 0$, $\nexists x \in \mathbf{R}$ such that $f(x) = y$

5. $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = 3x^3 + 2$, is one to one.

   Proof. Suppose that $f(x_1) = f(x_2)$

   $\Rightarrow 3x_1^3 + 2 = 3x_2^3 + 2$

   $\Rightarrow 3x_1^3 = 3x_2^3$

   $\Rightarrow x_1^3 = x_2^3$

   $\Rightarrow x_1 = x_2$

   i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

   Hence, $f(x)$ is one to one. ■

6. $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = 3x^3 + 2$, is onto.

Proof. We must show that given $y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ such that $f(x) = y$

So let $y \in \mathbf{R}$ be given.

Let $x = \underline{\sqrt[3]{\frac{y-2}{3}}}$

Observe: $f(x) = 3x^3 + 2 = \underline{3\left(\sqrt[3]{\frac{y-2}{3}}\right)^3 + 2} = \underline{3\left(\frac{y-2}{3}\right) + 2} = \underline{(y-2) + 2} = y$

i.e., Given $y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ (namely $x = \underline{\sqrt[3]{\frac{y-2}{3}}}$) such that $f(x) = y$.

Hence, $f(x)$ is one to one. ■

Scratch Work We want x such that $f(x) = y$

$\Rightarrow 3x^3 + 2 = y$

$\Rightarrow 3x^3 = y - 2$

$\Rightarrow x^3 = \frac{y-2}{3}$

$\Rightarrow x = \sqrt[3]{\frac{y-2}{3}}$