MTH 1126 Test #3 - Solutions

FALL 2015

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Show CLEARLY how you arrive at your answers.

1. $\int x \sin(x)\, dx =$

Use Integration by Parts:

$$\int x \sin(x)\ dx = \int \underbrace{x}_{u} \underbrace{\sin(x)\ dx}_{dv} = \int u dv = uv - \int v du = \underbrace{x}_{u} \cdot \underbrace{(-\cos(x))}_{v} - \int \underbrace{(-\cos(x))}_{v} \cdot \underbrace{dx}_{du}$$

$$= -x\cos(x) + \int \cos(x)\, dx = -x\cos(x) + \sin(x) + C$$

u	$=$	x	dv	$=$	$\sin(x)\, dx$
$\frac{du}{dx}$	$=$	1	$\int dv$	$=$	$\int \sin(x)\, dx$
du	$=$	dx	v	$=$	$-\cos(x)$

i.e. $\int x\sin(x)\ dx = -x\cos(x) + \sin(x) + C$

2. $\lim_{x\to 0} \frac{\sin(2x)}{x} \sim \frac{0}{0}$ So use L'Hôpital's Rule.

$\lim_{x\to 0} \frac{\sin(2x)}{x} = \lim_{x\to 0} \frac{2\cos(2x)}{1} = 2$

i.e., $\lim_{x\to 0} \frac{\sin(2x)}{x} = 2$

3. $\int \frac{7x+11}{x^2+x-6} dx =$

Express as $\int \frac{7x+11}{(x+3)(x-2)} dx$

Observe:

$\frac{7x+11}{(x+3)(x-2)} = \frac{C_1}{x+3} + \frac{C_2}{x-2}$

$\Rightarrow 7x + 11 = C_1(x-2) + C_2(x+3)$

Solve for the constants, by plugging in "strategic values" of x.

$x = 2$ We have: $25 = 5C_2 \Rightarrow \mathbf{C}_2 = 5$

$x = -3$ We have: $-10 = -5C_1 \Rightarrow \mathbf{C}_1 = \mathbf{2}$.

Therefore:

$\int \frac{7x+11}{x^2+x-6} dx = \int \left(\frac{2}{x+3} + \frac{5}{x-2}\right) dx = \int \frac{2}{x+3} dx + \int \frac{5}{x-2} dx = 2\ln|x+3| + 5\ln|x-2| + C$

i.e., $\int \frac{7x+11}{x^2+x-6} dx = 2\ln|x+3| + 5\ln|x-2| + C$

4. $\int \sin^7(x)\cos^3(x)\,dx =$

(sine to an odd power) Pull out a factor of $\sin(x)$ to serve as the "future du", and convert the rest of the sines to cosines using the identity $\sin^2(x) = 1 - \cos^2(x)$.

$$\int \sin^7(x)\cos^3(x)\,dx = \int \sin^6(x)\cos^3(x)\underbrace{\sin(x)\;dx}_{\text{future } du} = \int \left(\sin^2(x)\right)^3 \cos^3(x)\sin(x)\,dx$$

$$= \int \left(1 - \cos^2(x)\right)^3 \cos^3(x)\sin(x)\,dx.$$

Let $u = \cos(x)$

$\Rightarrow du = -\sin(x)\,dx$

$\Rightarrow -du = \sin(x)\,dx$

Continuing, we have:

$$\int \underbrace{\left(1-\cos^2(x)\right)^3}_{(1-u^2)^3}\underbrace{\cos^3(x)}_{u^3}\underbrace{\sin(x)\,dx}_{-du} = \int \left(1-u^2\right)^3 u^3\,(-du) = \int \left(-u^6 + 3u^4 - 3u^2 + 1\right) u^3\,(-du)$$

$$= \int \left(u^6 - 3u^4 + 3u^2 - 1\right) u^3 du = \int \left(u^9 - 3u^7 + 3u^5 - u^3\right) du =$$

$$= \tfrac{1}{10}u^{10} - \tfrac{3}{8}u^8 + \tfrac{1}{2}u^6 - \tfrac{1}{4}u^4 + C$$

$$= \tfrac{1}{10}\cos^{10}(x) - \tfrac{3}{8}\cos^8(x) + \tfrac{1}{2}\cos^6(x) - \tfrac{1}{4}\cos^4(x) + C$$

$$\boxed{\text{i.e., } \int \sin^7(x)\cos^3(x)\,dx = \tfrac{1}{10}\cos^{10}(x) - \tfrac{3}{8}\cos^8(x) + \tfrac{1}{2}\cos^6(x) - \tfrac{1}{4}\cos^4(x) + C}$$

Alternatively:

$\int \sin^7(x)\cos^3(x)\,dx =$

(cosine to an odd power) Pull out a factor of $\cos(x)$ to serve as the "future du", and convert the rest of the cosines into sines using the identity $\cos^2(x) = 1 - \sin^2(x)$.

$$\int \sin^7(x)\cos^3(x)\,dx = \int \sin^7(x)\cos^2(x)\underbrace{\cos(x)\;dx}_{\text{future } du} = \int \sin^7(x)\left(1 - \sin^2(x)\right)\cos(x)\,dx$$

Let $u = \sin(x)$

$\Rightarrow du = \cos(x)\,dx$

Continuing, we have:

$$\int \underbrace{\sin^7(x)}_{u^7}\underbrace{\left(1-\sin^2(x)\right)}_{(1-u^2)}\underbrace{\cos(x)\,dx}_{du} = \int u^7\left(1-u^2\right)(du) = \int \left(u^7 - u^9\right)(du)$$

$$= \tfrac{1}{8}u^8 - \tfrac{1}{10}u^{10} + C = \tfrac{1}{8}\sin^8(x) - \tfrac{1}{10}\sin^{10}(x) + C$$

$$\boxed{\text{i.e., } \int \sin^7(x)\cos^3(x)\,dx = \tfrac{1}{8}\sin^8(x) - \tfrac{1}{10}\sin^{10}(x) + C}$$

5. $\int \frac{1}{\sqrt{4+9x^2}} dx =$

Use trig substitution to get rid of the radical. We want to replace $4+9x^2$ with something of the form $a^2 + a^2 \tan^2(\theta)$.

$a^2 = 4$

$\Rightarrow a = 2$

Also:

$9x^2 = a^2 \tan^2(\theta)$

$\Rightarrow 9x^2 = 4\tan^2(\theta)$

$\Rightarrow 3x = 2\tan(\theta)$

$\Rightarrow x = \frac{2}{3}\tan(\theta)$

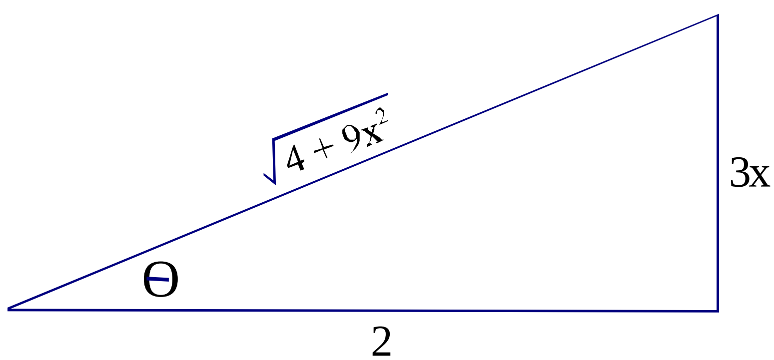
$\Rightarrow dx = \frac{2}{3}\sec^2(\theta)\, d\theta$

Thus, we have:

$$\int \frac{1}{\sqrt{4+9x^2}} dx = \int \frac{1}{\sqrt{4+4\tan^2(\theta)}} \frac{2}{3}\sec^2(\theta)\, d\theta = \int \frac{1}{\sqrt{4\sec^2(\theta)}} \frac{2}{3}\sec^2(\theta)\, d\theta = \int \frac{1}{2\sec(\theta)} \frac{2}{3}\sec^2(\theta)\, d\theta$$

$$= \frac{1}{3}\int \sec(\theta)\, d\theta = \frac{1}{3}\ln|\sec(\theta)+\tan(\theta)| + C$$

Observe: $x = \frac{2}{3}\tan(\theta) \Rightarrow \frac{3x}{2} = \tan(\theta) = \frac{\text{opp}}{\text{adj}}$. This yields the triangle below:



$$\Rightarrow \int \frac{1}{\sqrt{4+9x^2}} dx = \frac{1}{3}\ln|\sec(\theta)+\tan(\theta)| + C = \frac{1}{3}\ln\left|\frac{\sqrt{4+9x^2}}{2} + \frac{3x}{2}\right| + C$$

Simplifying, we have: $\frac{1}{3}\ln\left|\left(\frac{1}{2}\right)\left(\sqrt{4+9x^2}+3x\right)\right| + C = \frac{1}{3}\left(\ln\left(\frac{1}{2}\right) + \ln\left|\sqrt{4+9x^2}+3x\right|\right) + C$

$$= \underbrace{\frac{1}{3}\ln\left(\frac{1}{2}\right)}_{\text{Constant}} + \frac{1}{3}\ln\left|\sqrt{4+9x^2}+3x\right| + C = \frac{1}{3}\ln\left|\sqrt{4+9x^2}+3x\right| + C_1$$

$$\boxed{\text{i.e., } \int \frac{1}{\sqrt{4+9x^2}} dx = \frac{1}{3}\ln\left|\frac{\sqrt{4+9x^2}}{2} + \frac{3x}{2}\right| + C = \frac{1}{3}\ln\left|\sqrt{4+9x^2}+3x\right| + C_1}$$

Choose one exercise from exercises 6 and 7.

6. $\int \frac{x-18}{(x^2+4)(x-2)} dx =$

   Let's decompose this using partial fractions.

   $\frac{x-18}{(x^2+4)(x-2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2}$

   Remark 1 *Note that since x^2+4 is an* ***irreducible*** *quadratic, we put an $Ax+B$ above it.*

   We multiply both sides by the least common denominator and get

   $x - 18 = (Ax+B)(x-2) + C(x^2+4)$

   Solve for the constants, by plugging in "strategic values" of x.

   $\boxed{x=2}$ We have: $-16 = 8C \Rightarrow \boxed{C=-2}$

   This gives us:

   $x - 18 = (Ax+B)(x-2) - 2(x^2+4)$

   $\boxed{x=0}$ We have: $-18 = -2B - 8 \Rightarrow -2B = -10 \Rightarrow \boxed{B=5}$

   This gives us:

   $x - 18 = (Ax+5)(x-2) - 2(x^2+4)$

   There don't seem to be any other really "strategic values" of x to plug in. Soooo . . . we will multiply everything out and compare coefficients of x^2 on both sides.

   $x - 18 = (A-2)x^2 + (5-2A)x - 18$

   Comparing coefficients of x^2, we have: $A - 2 = 0$

   $\Rightarrow \boxed{A=2}$

   Thus, we have:

   $$\int \frac{x-18}{(x^2+4)(x-2)} dx = \int \left(\frac{Ax+B}{x^2+4} + \frac{C}{x-2}\right) dx = \int \left(\frac{2x+5}{x^2+4} - \frac{2}{x-2}\right) dx$$

   $$= \int \frac{2x+5}{x^2+4} dx - \int \frac{2}{x-2} dx = \int \frac{1}{x^2+4} 2x dx + 5\int \frac{1}{x^2+4} dx - \int \frac{2}{x-2} dx$$

   (Let $u = x^2+4$; then $du = 2x\,dx$.)

   $= \int \frac{1}{u} du + 5\int \frac{1}{x^2+4} dx - 2\int \frac{1}{x-2} dx = \ln|u| + 5\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{x}{2}\right) - 2\ln|x-2| + C$

   $= \ln\left|x^2+4\right| + \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) - 2\ln|x-2| + C$

   $\boxed{\text{i.e., } \int \frac{x-18}{(x^2+4)(x-2)} dx = \ln\left|x^2+4\right| + \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) - 2\ln|x-2| + C}$

7. $\int \sin^2(x)\cos^2(x)\,dx =$

(sine and cosine raised to even powers) Reduce the powers of sine and cosine by using the double angle formulas:

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \text{and} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

Continuing:

$$\int \sin^2(x)\cos^2(x)\,dx = \int \left(\tfrac{1-\cos(2x)}{2}\right)\left(\tfrac{1+\cos(2x)}{2}\right)dx = \tfrac{1}{4}\int \left(1-\cos^2(2x)\right)dx$$

$$= \tfrac{1}{4}\int \left(1-\tfrac{1+\cos(4x)}{2}\right)dx = \tfrac{1}{4}\int \left(1-\tfrac{1}{2}-\tfrac{1}{2}\cos(4x)\right)dx$$

$$= \tfrac{1}{4}\int \left(\tfrac{1}{2}-\tfrac{1}{2}\cos(4x)\right)dx = \tfrac{1}{8}\int (1-\cos(4x))\,dx$$

$$= \tfrac{1}{8}\left(x-\tfrac{1}{4}\sin(4x)\right)+C = \tfrac{1}{8}x-\tfrac{1}{32}\sin(4x)+C$$

i.e., $\int \sin^2(x)\cos^2(x)\,dx = \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$

8. $\int x\ln(x)\,dx =$

Use Integration by Parts:

$$\int x\ln(x)\;dx = \int \underbrace{\ln(x)}_{u}\underbrace{x\,dx}_{dv} = \int u\,dv = uv - \int v\,du = \underbrace{\ln(x)}_{u}\cdot\underbrace{\frac{1}{2}x^2}_{v} - \int \underbrace{\frac{1}{2}x^2}_{v}\cdot\underbrace{\frac{1}{x}dx}_{du}$$

$$= \tfrac{1}{2}x^2\ln(x) - \tfrac{1}{2}\int x\,dx = \tfrac{1}{2}x^2\ln(x) - \tfrac{1}{2}\tfrac{x^2}{2} + C = \tfrac{1}{2}x^2\ln(x) - \tfrac{1}{4}x^2 + C$$

u	$=$	$\ln(x)$	dv	$=$	$x\,dx$
$\frac{du}{dx}$	$=$	$\frac{1}{x}$	$\int dv$	$=$	$\int x\,dx$
du	$=$	$\frac{1}{x}dx$	v	$=$	$\frac{1}{2}x^2$

i.e. $\int x\ln(x)\;dx = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C$

9. $\lim_{x\to 0^+} x^2\ln(x) \sim 0\cdot(-\infty)$

Re-express as $\lim_{x\to 0^+} \frac{\ln(x)}{x^{-2}} \sim \frac{-\infty}{\infty}$ So we can use L'Hôpital's Rule.

$$\lim_{x\to 0^+} x^2\ln(x) = \lim_{x\to 0^+}\frac{\ln(x)}{x^{-2}} = \lim_{x\to 0^+}\frac{\left(\frac{1}{x}\right)}{-2x^{-3}} = \lim_{x\to 0^+}\frac{\left(\frac{1}{x}\right)}{-2x^{-3}}\cdot\frac{x^3}{x^3} = \lim_{x\to 0^+} -\tfrac{1}{2}x^2 = 0$$

i.e., $\lim_{x\to 0^+} x^2\ln(x) = 0$