MTH 3311 – Test #2 – Solutions

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Directions: Do two of the three exercises.

1. A paratrooper and parachute weigh 240 lb. At the instant the parachute opens, he is traveling vertically downward at 40 $\frac{\text{ft}}{\text{sec}}$. If the air resistance varies directly as the instantaneous velocity, and the air resistance is 80 lb when the velocity is 20 $\frac{\text{ft}}{\text{sec}}$:

   a) Find the limiting velocity

   b) Determine the position and velocity at any time t.

First, we establish our conventions regarding direction:

Positive Direction $\uparrow$

We will start our stopwatch at the instant that the parachute opens.

Thus, $v\,(0 \text{ sec}) = -40\ \frac{\text{ft}}{\text{sec}}$ (The velocity is negative because the motion is in the downward direction.)

Let R be the resistance due to air.

When $v = -20\frac{\text{ft}}{\text{sec}}$, $R = 80$ lb   (Because "For a velocity of $20\frac{\text{ft}}{\text{sec}}$ (in the *negative* direction), ... air resistance is 80 lb.")

Also: "The force of air resistance is proportional to the velocity" i.e. $R \propto v$

$\Rightarrow R = kv$, where k is the **constant of proportionality.**

For "future reference," we will find the constant of proportionality right now.

Recall: When $v = -20\frac{\text{ft}}{\text{sec}}$, $R = 80$ lb

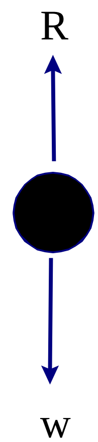
Also: $R = kv$

$\Rightarrow R = 80 \text{ lb} = k\left(-20\frac{\text{ft}}{\text{sec}}\right)$

$\Rightarrow k = \frac{80 \text{ lb}}{-20\frac{\text{ft}}{\text{sec}}} = -4\frac{\text{lb sec}}{\text{ft}}$

$\Rightarrow k = -4\frac{\text{lb sec}}{\text{ft}}$ (For "future reference")

Next: Since there is more than one force acting on the object, let's draw a force diagram of the object.



From the force diagram, the total force $F = w + R$,

where w is the **weight** of the object and R is the force on the object, due to air resistance.

Remark: To allow ourselves to model this relationship as a differential equation, we will employ a **standard trick:**

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Note well: When more than one force is acting on a free falling object, our approach will usually be to set F (the sum or all forces on the object) equal to ma.

$$\underbrace{(\text{sum of all forces})}_{F} = \underbrace{ma}_{F}$$

This is a standard approach for velocity exercises!!!

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Our "Standard Trick" yields the equation $\underbrace{w+R}_{\text{Sum of all forces}} = \underbrace{ma}_{F}$ (Eq. 1)

Recall: acceleration is the derivative of velocity. i.e., $a = \frac{dv}{dt}$

Thus, Eq. 1 can be rendered:

$$\underbrace{w + kv}_{w+R} = \underbrace{m\frac{dv}{dt}}_{ma}$$

This is a differential equation in v.

Let's solve it!

$-m\frac{dv}{dt} + kv = -w$

$$\Rightarrow \frac{dv}{dt} + \underbrace{\left(-\frac{k}{m}\right)}_{P(t)} v = \underbrace{\frac{w}{m}}_{Q(t)}$$

Compute the integrating factor, $e^{\int P(t)dt} = e^{\int\left(-\frac{k}{m}\right)dt} = e^{-\frac{k}{m}t}$

Multiplying both sides by the integrating factor, we have:

$e^{-\frac{k}{m}t}\frac{dv}{dt} + \left(-\frac{k}{m}\right)e^{-\frac{k}{m}t}v = \frac{w}{m}e^{-\frac{k}{m}t}$

$\Rightarrow \frac{d}{dt}\left[e^{-\frac{k}{m}t}v\right] = \frac{w}{m}e^{-\frac{k}{m}t}$

$\Rightarrow \int\left(\frac{d}{dt}\left[e^{-\frac{k}{m}t}v\right]\right)dt = \int \frac{w}{m}e^{-\frac{k}{m}t}dt$

$\Rightarrow e^{-\frac{k}{m}t}v = \frac{w}{m}\left(-\frac{m}{k}\right)e^{-\frac{k}{m}t} = -\frac{w}{k}e^{-\frac{k}{m}t} + C$

i.e. $e^{-\frac{k}{m}t}v = -\frac{w}{k}e^{-\frac{k}{m}t} + C$

$\Rightarrow v = -\frac{w}{k} + e^{\frac{k}{m}t}C$

Now, let's find the constant C

Recall: $v\,(0 \text{ sec}) = -40\frac{\text{ft}}{\text{sec}}$

$\Rightarrow -40\frac{\text{ft}}{\text{sec}} = v\,(0 \text{ sec}) = -\frac{w}{k} + e^{\frac{k}{m}(0 \text{ sec})}C = -\frac{w}{k} + C$

i.e. $-40\frac{\text{ft}}{\text{sec}} = -\frac{w}{k} + C$

$\Rightarrow C = \frac{w}{k} - 40\frac{\text{ft}}{\text{sec}}$

$\Rightarrow v = -\frac{w}{k} + \left(\frac{w}{k} - 40\frac{\text{ft}}{\text{sec}}\right)e^{\frac{k}{m}t}$

To find $\frac{w}{k}$, recall two things:

First, $k = -4\frac{\text{lb sec}}{\text{ft}}$

Next, the weight, $w = -240$ lb.

Thus, $\frac{w}{k} = \frac{-240 \text{ lb}}{-4\frac{\text{lb sec}}{\text{ft}}} = 60\ \frac{\text{ft}}{\text{sec}}*$

i.e., $\frac{w}{k} = 60\ \frac{\text{ft}}{\text{sec}}$

Finally, we want to find $\frac{k}{m}$.

Note that $w = mg$, where g is the acceleration due to gravity.

$\Rightarrow m = \frac{w}{g} = \frac{-240 \text{ lb}}{-32\frac{\text{ft}}{\text{sec}^2}} = 7.5\frac{\text{lb sec}^2}{\text{ft}}$

i.e., $m = 7.5\frac{\text{lb sec}^2}{\text{ft}}$

Hence, observe that $\frac{k}{m} = \frac{-4\frac{\text{lb sec}}{\text{ft}}}{7.5\frac{\text{lb sec}^2}{\text{ft}}} = -\frac{0.533}{\text{sec}}$

Therefore, **velocity** is given by: $v\,(t) = -60\frac{\text{ft}}{\text{sec}} + \left(60\frac{\text{ft}}{\text{sec}} - 40\frac{\text{ft}}{\text{sec}}\right)e^{-\frac{0.533}{\text{sec}}t} = -60\frac{\text{ft}}{\text{sec}} + 20\frac{\text{ft}}{\text{sec}}e^{-\frac{0.533}{\text{sec}}t}$

b) Therefore, **velocity** is given by: $v\,(t) = -60\frac{\text{ft}}{\text{sec}} + 20\frac{\text{ft}}{\text{sec}}e^{-\frac{0.533}{\text{sec}}t}$

a) Find the limiting velocity

limiting velocity $= \lim_{t\to\infty} v\,(t) = \lim_{t\to\infty}\left(v\,(t) = -60\frac{\text{ft}}{\text{sec}} + 20\frac{\text{ft}}{\text{sec}}e^{-\frac{0.533}{\text{sec}}t}\right) = -60\frac{\text{ft}}{\text{sec}}$

The **limiting velocity** is given by: $v\,(t) = -60\frac{\text{ft}}{\text{sec}}$

Ouch! At THAT speed, it's really gonna hurt when he hits the ground!!!

To find the **position** at time t, recall that the vertical position $s = \int v\,(t)\,dt$

$s(t) = \int v(t)\,dt = \int \left(-60\frac{\text{ft}}{\text{sec}} + 20\frac{\text{ft}}{\text{sec}}e^{-\frac{0.533}{\text{sec}}t}\right) dt = -60\frac{\text{ft}}{\text{sec}}t + 20\frac{\text{ft}}{\text{sec}}\left(-\frac{1}{0.533}\text{ sec}\right)e^{-\frac{0.533}{\text{sec}}t} + C$

$= -60\frac{\text{ft}}{\text{sec}}t - \frac{20}{0.533}\text{ ft } e^{-\frac{0.533}{\text{sec}}t} + C$

i.e., $s(t) = -60\frac{\text{ft}}{\text{sec}}t - \frac{20}{0.533}\text{ ft } e^{-\frac{0.533}{\text{sec}}t} + C$

No "initial position" is given in the problem, so we will assume that the initial position is 0 ft

Thus, $0\text{ ft} = s(0\text{ sec}) = -60\frac{\text{ft}}{\text{sec}}(0\text{ sec}) - \frac{20}{0.533}\text{ ft } e^{-\frac{0.533}{\text{sec}}(0\text{ sec})} + C$

i.e., $0\text{ ft} = -37.523\text{ ft } + C$

$\Rightarrow C = 37.523\text{ ft}$

Thus, the vertical position is given by: $s(t) = -60\frac{\text{ft}}{\text{sec}}t - \frac{20}{0.533}fte^{-\frac{0.533}{\text{sec}}t} + 37.523\text{ ft}$

2. Water at 90 °C cools in 20 minutes to 75 °C in a room of temperature of 25 °C.

   a) Find the temperature of the water after 30 minutes

   b) When is the temperature 50 °C ?

Let $T =$ temperature

$t =$ time

$r =$ room temperature $= 25\,^{\circ}\mathrm{C}$

According to **Newton's Law of Cooling,** the rate of change of water temperature, with respect to time, is proportional to the **difference** between the air and water temperatures.

Hence, $\frac{dT}{dt} = k\left(T - r\right)$ (Where k is the constant of proportionality)

$\Rightarrow \frac{dT}{(T-r)} = kdt$

$\Rightarrow \int \frac{1}{(T-r)} dT = \int kdt$

$\Rightarrow \ln\left(T - r\right) = kt + C$ (We can assume that $T - r > 0$, since $T > r$ initially. Hence, we don't need absolute value bars.)

$\Rightarrow e^{\ln(T-r)} = e^{kt+C} = C_1 e^{kt}$

i.e., $T - r = C_1 e^{kt}$

(Since $r = 25\,^{\circ}\mathrm{C}$ is constant, we put the value in here.)

i.e., $T - 25\,^{\circ}\mathrm{C} = C_1 e^{kt}$

$\Rightarrow T = 25\,^{\circ}\mathrm{C} + C_1 e^{kt}$

Notice that we have two constants to evaluate.

Therefore, we need two initial conditions

Recall: at $t = 0\,\mathrm{min}$, $T = 90\,^{\circ}\mathrm{C}$

$\Rightarrow 90\,^{\circ}\mathrm{C} = T\left(0\,\mathrm{min}\right) = 25\,^{\circ}\mathrm{C} + C_1 e^{k(0\,\mathrm{min})}$

$\Rightarrow 90\,^{\circ}\mathrm{C} = 25\,^{\circ}\mathrm{C} + C_1$

$\Rightarrow 65\,^{\circ}\mathrm{C} = C_1$

Thus, $T\left(t\right) = 25\,^{\circ}\mathrm{C} + 65\,^{\circ}\mathrm{C} e^{kt}$

Recall Also: at $t = 20\,\mathrm{min}$, $T = 75\,^{\circ}\mathrm{C}$

This yields: $75\,^{\circ}\mathrm{C} = T\left(20\,\mathrm{min}\right) = 25\,^{\circ}\mathrm{C} + 65\,^{\circ}\mathrm{C} e^{k(20\,\mathrm{min})}$

i.e., $75\,^{\circ}\mathrm{C} = 25\,^{\circ}\mathrm{C} + 65\,^{\circ}\mathrm{C} e^{k(20\,\mathrm{min})}$

$\Rightarrow 50\,^{\circ}\mathrm{C} = 65\,^{\circ}\mathrm{C} e^{k(20\,\mathrm{min})}$

$\Rightarrow \frac{50\,^{\circ}\mathrm{C}}{65\,^{\circ}\mathrm{C}} = e^{k(20\,\mathrm{min})}$

$\Rightarrow \ln\left(\frac{50\,^{\circ}\mathrm{C}}{65\,^{\circ}\mathrm{C}}\right) = \ln\left(e^{k(20\,\mathrm{min})}\right)$

$\Rightarrow \ln\left(\frac{50\,^{\circ}\mathrm{C}}{65\,^{\circ}\mathrm{C}}\right) = k\left(20\,\mathrm{min}\right)$

$\Rightarrow \frac{\ln\left(\frac{50\,^{\circ}\mathrm{C}}{65\,^{\circ}\mathrm{C}}\right)}{(20\,\mathrm{min})} = k$

$\Rightarrow k = -\frac{0.0131}{\text{min}}$

$$\boxed{\Rightarrow T(t) = 25\,°\text{C} + 65\,°\text{C}e^{-\frac{0.0131}{\text{min}}t}}$$

a) Find the temperature of the water after 30 minutes

$T(30\,\text{min}) = 25\,°\text{C} + 65\,°\text{C}e^{-\frac{0.0131}{\text{min}}(30\text{ min})} = 68.877\,°\text{C}$

$$\boxed{T(30\,\text{min}) = 68.877\,°\text{C}}$$

b) When is the temperature 50 °C ?

$\Rightarrow 50\,°\text{C} = T(t) = 25\,°\text{C} + 65\,°\text{C}e^{-\frac{0.0131}{\text{min}}t}$

i.e., $50\,°\text{C} = 25\,°\text{C} + 65\,°\text{C}e^{-\frac{0.0131}{\text{min}}t}$

$\Rightarrow 25\,°\text{C} = 65\,°\text{C}e^{-\frac{0.0131}{\text{min}}t}$

$\Rightarrow \frac{25}{65} = e^{-\frac{0.0131}{\text{min}}t}$

$\Rightarrow \ln\left(\frac{25}{65}\right) = \ln\left(e^{-\frac{0.0131}{\text{min}}t}\right)$

$\Rightarrow \ln\left(\frac{25}{65}\right) = -\frac{0.0131}{\text{min}}t$

$\Rightarrow t = \ln\left(\frac{25}{65}\right)\left(-\frac{\text{min}}{0.0131}\right) = 72.94\,\text{min}$

$$\boxed{\text{i.e.,} \Rightarrow t = 72.94\,\text{min when } T = 50\ °\text{C}}$$

3. The demand and supply of a certain commodity are given in terms of thousands of units, respectively, by

$$D = 50 + 7p(t) + 2p'(t); \quad S = 350 - 8p(t) - 3p'(t).$$

At $t = 0$, the price of the commodity is 35 units.

a) Find the price at any later time and obtain its graph.

b) determine whether there is price stability and the equilibrium price if one exists.

Equating supply and demand, we have:

$50 + 7p(t) + 2p'(t) = 350 - 8p(t) - 3p'(t)$

$\Rightarrow 5p'(t) + 15p(t) = 300$

$\Rightarrow p'(t) + \underbrace{3}_{P(t)} p(t) = \underbrace{60}_{Q(t)}$

Our integrating factor is $e^{\int P(t)dt} = e^{\int 3dt} = e^{3t}$

Multiplying both sides by the integrating factor, e^{3t}, we have:

$e^{3t}p'(t) + 3e^{3t}p(t) = 60e^{3t}$

$\Rightarrow \frac{d}{dt}\left[e^{3t}p(t)\right] = 60e^{3t}$ Integrating, we have:

$\Rightarrow \int \left(\frac{d}{dt}\left[e^{3t}p(t)\right]\right) dt = \int 60e^{3t}dt$

$\Rightarrow e^{3t}p(t) = 60\left(\frac{1}{3}\right)e^{3t} + C = 20e^{3t} + C$

i.e., $e^{3t}p(t) = 20e^{3t} + C$

$\Rightarrow p(t) = 20 + e^{-3t}C$

To find the constant C, we use our initial condition $p(0) = 35$ (Because "At $t = 0$, the price of the commodity is 35 units.")

$\Rightarrow 35 = p(0) = 20 + e^{-3(0)}C = 20 + C$

i.e., $35 = 20 + C$

i.e., $15 = C$

Hence, $p(t) = 20 + 15e^{-3t}$ is the price at any time t.

To graph the function, let's consider the derivative.

$p'(t) = -45e^{-3t}$

Note that $p'(t) < 0$ for all values of t, since $e^{\text{ham sandwich}}$ is always positive.

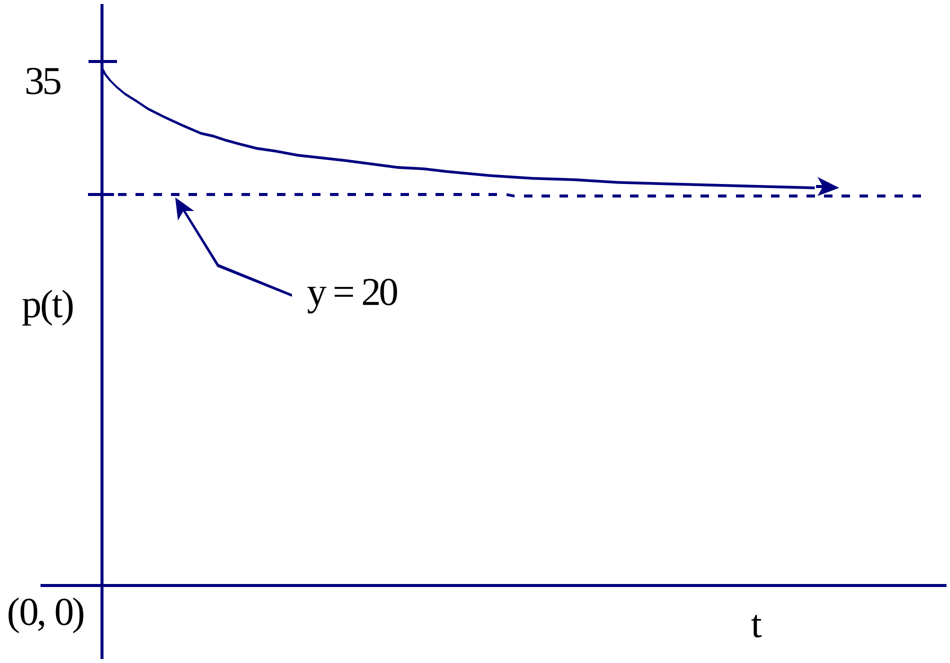
Hence, the graph of $p(t)$ is decreasing.

Next, let's consider the graph of $p(t)$ as $t \to \infty$.

$\lim_{t\to\infty} p(t) = \lim_{t\to\infty}\left(20 + 15e^{-3t}\right) = 20 + 0 = 20$

i.e., $\lim_{t\to\infty} p(t) = 20$

The graph of $y = p(t)$ is given below:



The market is **stable.** The equilibrium price is 20 units. The price will continue to decrease toward the equilibrium price $p_e = 20$.