MTH 3311 - Test #3 - Solutions

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Instructions: Show CLEARLY how you arrive at your answers

Do Exercise #1

1. Solve the Differential Equation: $y'' - 6y' + 8y = 52\sin(3x) - 39\cos(3x)$

   First, we find the Complementary Solution (i.e., the solution of the related equation: $y'' - 6y' + 8y = 0$)

   $y'' - 6y' + 8y = 0$

   $\Rightarrow \underbrace{\left(D^2 - 6D + 8\right)}_{\phi(D)} y = 0$

   $\Rightarrow \underbrace{m^2 - 6m + 8}_{\phi(m)} = 0$ (This is our auxiliary equation)

   $\Rightarrow (m-2)(m-4) = 0$

   $\Rightarrow m_1 = 2$ and $m_2 = 4$

   $\Rightarrow y_c = C_1 e^{2x} + C_2 e^{4x}$ (This is our complementary solution)

   Next, we find the particular solution.

   Since the right hand side of the equation is a linear combination of $\sin(3x)$ and $\cos(3x)$, we can use the Method of Undetermined Coefficients.

   Since the right hand side of the equation is a linear combination of $\sin(3x)$ and $\cos(3x)$, we suspect that the particular solution is of the form: $y_p = C_1\sin(3x) + C_2\cos(3x)$

   Next, we'll compute y_p' and y_p'' and plug y_p, y_p', and y_p'' into the equation:

   $y'' - 6y' + 8y = 52\sin(3x) - 39\cos(3x)$ and solve for $C_1$and C_2.

   (Continued on the next page)

$y_p = C_1 \sin(3x) + C_2 \cos(3x)$

$y_p' = 3C_1 \cos(3x) - 3C_2 \sin(3x)$

$y_p'' = -9C_1 \sin(3x) - 9C_2 \cos(3x)$

$$\begin{aligned} y_p'' - 6y_p' + 8y_p &= (-9C_1 \sin(3x) - 9C_2 \cos(3x)) \\ &- 6(3C_1 \cos(3x) - 3C_2 \sin(3x)) \\ &+ 8(C_1 \sin(3x) + C_2 \cos(3x)) \\ &= 52 \sin(3x) - 39 \cos(3x) \end{aligned}$$

Simplifying, we have:

$(18C_2 - C_1)\sin(3x) + (-18C_1 - C_2)\cos(3x) = 52\sin(3x) - 39\cos(3x)$

$\Rightarrow -C_1 + 18C_2 = 52$ (Eq. 1)
and
$-18C_1 - C_2 = -39$ (Eq. 2)

From Eq. 1, we get: $C_1 = 18C_2 - 52$

Plugging this into Eq. 2 for C_1, we get:
$-18(18C_2 - 52) - C_2 = -39$
$\Rightarrow -325C_2 = -975$
$\Rightarrow C_2 = 3$

Plugging $C_2 = 3$ into Eq. 2, we get:
$\Rightarrow -18C_1 - 3 = -39$
$\Rightarrow -18C_1 = -36$
$\Rightarrow C_1 = 2$

Our Particular Solution is $y_p = C_1 \sin(3x) + C_2 \cos(3x) = 2\sin(3x) + 3\cos(3x)$

Our General Solution is: $y = \underbrace{2\sin(3x) + 3\cos(3x)}_{y_p} + \underbrace{C_1 e^{2x} + C_2 e^{4x}}_{y_c}$

$$\boxed{y = 2\sin(3x) + 3\cos(3x) + C_1 e^{2x} + C_2 e^{4x}}$$

From Exercises 2-3, do one.

2. Solve the Differential Equation: $5y'' - 6y' + 5y = 15x^2 - 61x + 70$

   First, we find the Complementary Solution (i.e., the solution of the related equation: $5y'' - 6y' + 5y = 0$)

   $5y'' - 6y' + 5y = 0$

   $\Rightarrow \underbrace{\left(5D^2 - 6D + 5\right)}_{\phi(D)} y = 0$

   $\Rightarrow \underbrace{5m^2 - 6m + 5}_{\phi(m)} = 0$ (This is our auxiliary equation)

   $\Rightarrow m = \frac{-(-6)\pm\sqrt{(-6)^2-4(5)(5)}}{2(5)} = \frac{6\pm\sqrt{-64}}{10} = \frac{6\pm 8i}{10} = \frac{3}{5} \pm \frac{4}{5}i$

   $\Rightarrow m_1 = \frac{3}{5} + \frac{4}{5}i$ and $m_2 = \frac{3}{5} - \frac{4}{5}i$

   $\Rightarrow y_c = C_1 e^{\left(\frac{3}{5}+\frac{4}{5}i\right)x} + C_2 e^{\left(\frac{3}{5}-\frac{4}{5}i\right)x}$ (This is our complementary solution)

   But - we should rewrite it without using complex coefficients.

   $y_c = C_1 e^{\left(\frac{3}{5}+\frac{4}{5}i\right)x} + C_2 e^{\left(\frac{3}{5}-\frac{4}{5}i\right)x} = C_1 e^{\frac{3}{5}x+\frac{4}{5}ix} + C_2 e^{\frac{3}{5}x-\frac{4}{5}ix} = C_1 e^{\frac{3}{5}x} e^{i\frac{4}{5}x} + C_2 e^{\frac{3}{5}x} e^{-i\frac{4}{5}x}$

   $= e^{\frac{3}{5}x}\left(C_1 e^{i\frac{4}{5}x} + C_2 e^{-i\frac{4}{5}x}\right) = e^{\frac{3}{5}x}\left(A\cos\left(\frac{4}{5}x\right) + B\sin\left(\frac{4}{5}x\right)\right)$

   i.e. $y_c = e^{\frac{3}{5}x}\left(A\cos\left(\frac{4}{5}x\right) + B\sin\left(\frac{4}{5}x\right)\right)$ (This is our complementary solution)

   Remark: We used the identity: $C_1 e^{im_1 x} + C_2 e^{-im_2 x} = \left(A\cos(m_1 x) + B\sin(m_2 x)\right)$

   Next, we find the particular solution.

   Since the right hand side of the equation is a polynomial, we can use the Method of Undetermined Coefficients.

   Since the right hand side of the equation is $15x^2 - 61x + 70$, we suspect that the particular solution is of the form: $y_p = C_1 x^2 + C_2 x + C_3$

Next, we'll compute y_p' and y_p'' and plug y_p, y_p', and y_p'' into the equation $5y''-8y'+5y = 15x^2-61x+70$ to solve for C.

$y_p = C_1x^2 + C_2x + C_3$

$y_p' = 2C_1x + C_2$

$y_p'' = 2C_1$

$5y'' - 6y' + 5y = 5\left(2C_1\right) - 6\left(2C_1x + C_2\right) + 5\left(C_1x^2 + C_2x + C_3\right) = 15x^2 - 61x + 70$

$\Rightarrow 5C_1x^2 + \left(-12C_1 + 5C_2\right)x + \left(10C_1 - 6C_2 + 5C_3\right) = 15x^2 - 61x + 70$

Comparing coefficients of x, we have:

$5C_1 = 15 \Rightarrow C_1 = 3$

$$\boxed{C_1 = 3}$$

$-12C_1 + 5C_2 = -61 \Rightarrow -12\left(3\right) + 5C_2 = -61 \Rightarrow 5C_2 = -25 \Rightarrow C_2 = -5$

$$\boxed{C_2 = -5}$$

$\left(10C_1 - 6C_2 + 5C_3\right) = 70 \Rightarrow \left(10\left(3\right) - 6\left(-5\right) + 5C_3\right) = 10 \Rightarrow 5C_3 = 10 \Rightarrow C_3 = 2$

$$\boxed{C_3 = 2}$$

Our Particular Solution is $y_p = C_1x^2 + C_2x + C_3 = 3x^2 - 5x + 2$

i.e., $y_p = 3x^2 - 5x + 2$

Our General Solution is: $y = \underbrace{3x^2 - 5x + 2}_{y_p} + \underbrace{e^{\frac{3}{5}x}\left(A\cos\left(\frac{4}{5}x\right) + B\sin\left(\frac{4}{5}x\right)\right)}_{y_c}$

$$\boxed{y = 3x^2 - 5x + 2 + e^{\frac{3}{5}x}\left(A\cos\left(\tfrac{4}{5}x\right) + B\sin\left(\tfrac{4}{5}x\right)\right)}$$

3. Solve the Differential Equation: $y'' - 4y' - 12y = \ln(3x)$

First, we find the Complementary Solution (i.e., the solution of the related equation: $y'' - 4y' - 12y = 0$)

$y'' - 4y' - 12y = 0$

$\Rightarrow \underbrace{\left(D^2 - 4D - 12\right)}_{\phi(D)} y = 0$

$\Rightarrow \underbrace{m^2 - 4m - 12}_{\phi(m)} = 0$ (This is our auxiliary equation)

$\Rightarrow (m+2)(m-6) = 0$

$\Rightarrow m_1 = -2$ and $m_2 = 6$

$\Rightarrow y_c = C_1 e^{-2x} + C_2 e^{6x}$ (This is our complementary solution)

Next, we find the particular solution.

Since the right hand side of the equation is NOT a linear combination of polynomials, exponentials, or sines and cosines, we CAN'T use the Method of Undetermined Coefficients.

We have to use Variation of Parameters.

This means that we must vary the parameters C_1 and C_2 by converting them to functions of x. Namely, $A(x)$ and $B(x)$.

Our proposed general solution is:

$y = A(x) e^{-2x} + B(x) e^{6x}$

We impose two restrictions on the pair of functions $A(x)$ and $B(x)$.

The first restriction is that the pair $A(x)$ and $B(x)$ are such that $y = A(x) e^{-2x} + B(x) e^{6x}$ is the general solution of our equation.

There are infinitely many pairs of functions that satisfy this condition.

The second restriction, which we hold in abeyance for the time being, will uniquely define the pair $A(x)$ and $B(x)$.

Given $y = A(x)e^{-2x} + B(x)e^{6x}$, we compute y' and y''.

$y = A(x)e^{-2x} + B(x)e^{6x}$

$y' = A'(x)e^{-2x} + A(x)(-2e^{-2x}) + B'(x)e^{6x} + B(x)(6e^{6x})$

i.e., $y' = A'(x)e^{-2x} - 2A(x)e^{-2x} + B'(x)e^{6x} + 6B(x)e^{6x}$

We now impose our second restriction – namely that $A(x)$ and $B(x)$ are such that

$A'(x)e^{-2x} + B'(x)e^{6x} = 0.$

This restriction has the advantage that $A'(x)$ and $B'(x)$ are eliminated from the definition of y'

Thus, we have:

$y = A(x)e^{-2x} + B(x)e^{6x}$

$y' = -2A(x)e^{-2x} + 6B(x)e^{6x}$

$y'' = -2(A'(x)e^{-2x} + A(x)(-2e^{-2x})) + 6(B'(x)e^{6x} + B(x)(6e^{6x}))$

i.e., $y'' = -2A'(x)e^{-2x} + 4A(x)e^{-2x} + 6B'(x)e^{6x} + 36B(x)e^{6x}$

We can apply our second restriction again.

Recall: $A'(x)e^{-2x} + B'(x)e^{6x} = 0 \Rightarrow A'(x)e^{-2x} = -B'(x)e^{6x}$

Thus, $y'' = -2A'(x)e^{-2x} + 4A(x)e^{-2x} + 6B'(x)e^{6x} + 36B(x)e^{6x}$

Becomes: $y'' = -2(-B'(x)e^{6x}) + 4A(x)e^{-2x} + 6B'(x)e^{6x} + 36B(x)e^{6x}$

$\Rightarrow y'' = 4A(x)e^{-2x} + 8B'(x)e^{6x} + 36B(x)e^{6x}$

Next, we plug these definitions of y, y', y'' into the original equation: $y'' - 4y' - 12y = \ln(3x)$

$$\begin{array}{rcrlrlrlcl}
y'' & = & & 4A(x)e^{-2x} & + & 36B(x)e^{6x} & + & 8B'(x)e^{6x} & & \\
-4y' & = & + & 8A(x)e^{-2x} & - & 24B(x)e^{6x} & & & & \\
-12y & = & - & 12A(x)e^{-2x} & - & 12B(x)e^{6x} & & & & \\
\hline
y'' - 4y' - 12y & = & & & & & & 8B'(x)e^{6x} & = & \ln(3x)
\end{array}$$

$\Rightarrow 8B'(x)e^{6x} = \ln(3x) \Rightarrow B'(x) = \frac{1}{8}e^{-6x}\ln(3x)$

This exercise can't be taken any farther. If you made it this far, you received full credit.

Do Exercise 4.

4. Solve the Differential Equation: $x^2y'' - 4xy' + 6y = 6x^4 - 36x$

First, find the solution to the complementary equation $x^2y'' - 4xy' + 6y = 0$

Our strategy is to seek solutions of the form:

$y = x^\lambda$

$\Rightarrow y' = \lambda x^{\lambda-1}$

$\Rightarrow y'' = \lambda(\lambda - 1)x^{\lambda-2} = (\lambda^2 - \lambda)x^{\lambda-2}$

Plugging these into the complementary equation $x^2y'' - 4xy' + 6y = 0$, we have:

$x^2(\lambda^2 - \lambda)x^{\lambda-2} - 4x\lambda x^{\lambda-1} + 6x^\lambda = 0$

$\Rightarrow (\lambda^2 - \lambda)x^\lambda - 4\lambda x^\lambda + 6x^\lambda = 0$

$\Rightarrow (\lambda^2 - \lambda) - 4\lambda + 6 = 0$

$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$

$\Rightarrow (\lambda - 2)(\lambda - 3) = 0$

$\Rightarrow \lambda_1 = 2;\ \lambda_2 = 3$

Our complementary solution is:

$y_c = C_1x^{\lambda_1} + C_2x^{\lambda_2} = C_1x^2 + C_2x^3$

(Continued)

Next, we find our particular solution

Since the right hand side of the equation is the polynomial $6x^4 - 36x$, we guess that the particular solution is a polynomial having only terms of the same degree that appear on the right hand side of the original equation.

Thus, we guess that:

$$y = Ax^4 + Bx$$

$\Rightarrow y' = 4Ax^3 + B$

$\Rightarrow y'' = 12Ax^2$

To find A, and B, we plug these into the original equation, $x^2y'' - 4xy' + 6y = 6x^4 - 36x$.

This yields:

$$x^2\left(12Ax^2\right) - 4x\left(4Ax^3 + B\right) + 6\left(Ax^4 + Bx\right) = 6x^4 - 36x$$

$\Rightarrow 12Ax^4 - 16Ax^4 - 4Bx + 6Ax^4 + 6Bx = 6x^4 - 36x$

i.e., $2Ax^4 + 2Bx = 6x^4 - 36x$

$\Rightarrow 2A = 6 \Rightarrow A = 3$

Also: $2B = -36 \Rightarrow B = -18$

$\Rightarrow y_p = Ax^4 + Bx = 3x^4 - 18x$

The solution to the original equation is: $y = y_p + y_c$

The solution to the original equation is: $y_g = 3x^4 - 18x + C_1x^2 + C_2x^3$

Extra (WOW - 10 Points!)

Solve the Differential Equation: $x^2y'' - 4xy' + 6y = \ln(x)$

First, find the solution to the complementary equation $x^2y'' - 4xy' + 6y = 0$

Our strategy is to seek solutions of the form:

$$y = x^\lambda$$

$$\Rightarrow y' = \lambda x^{\lambda-1}$$

$$\Rightarrow y'' = \lambda(\lambda - 1)x^{\lambda-2} = \left(\lambda^2 - \lambda\right)x^{\lambda-2}$$

Plugging these into the complementary equation $x^2y'' - 4xy' + 6y = 0$, we have:

$$x^2\left(\lambda^2 - \lambda\right)x^{\lambda-2} - 4x\lambda x^{\lambda-1} + 6x^\lambda = 0$$

$$\Rightarrow \left(\lambda^2 - \lambda\right)x^\lambda - 4\lambda x^\lambda + 6x^\lambda = 0$$

$$\Rightarrow \left(\lambda^2 - \lambda\right) - 4\lambda + 6 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 3) = 0$$

$$\Rightarrow \lambda_1 = 2;\ \lambda_2 = 3$$

Our complementary solution is:

$$y_c = C_1x^{\lambda_1} + C_2x^{\lambda_2} = C_1x^2 + C_2x^3$$

(Continued)

Next, we find our particular solution

Since the right hand side of the equation is not a linear combination of powers of x, we **can't** use the Method of Undetermined Coefficients. Our only option is to use Variation of Parameters.

This means that we must vary the parameters C_1 and C_2 by converting them to functions of x. Namely, $A(x)$ and $B(x)$.

Our proposed general solution is:

$y = A(x)\,x^2 + B(x)\,x^3$

We impose two restrictions on the pair of functions $A(x)$ and $B(x)$.

The first restriction is that the pair $A(x)$ and $B(x)$ are such that $y = A(x)\,x^2 + B(x)\,x^3$ is the general solution of our equation.

There are infinitely many pairs of functions that satisfy this condition.

The second restriction, which we hold in abeyance for the time being, will uniquely define the pair $A(x)$ and $B(x)$.

Given $y = A(x)\,x^2 + B(x)\,x^3$, we compute y' and y''.

$y = A(x)\,x^2 + B(x)\,x^3$

$y' = A'(x)\,x^2 + A(x)\,(2x) + B'(x)\,x^3 + B(x)\,(3x^2)$

i.e., $y' = A'(x)\,x^2 + 2A(x)\,x + B'(x)\,x^3 + 3B(x)\,x^2$

We now impose our second restriction – namely that $A(x)$ and $B(x)$ are such that

$A'(x)\,x^2 + B'(x)\,x^3 = 0.$

This restriction has the advantage that $A'(x)$ and $B'(x)$ are eliminated from the definition of y'

Thus, we have:

$y = A(x)\,x^2 + B(x)\,x^3$

$y' = 2A(x)\,x + 3B(x)\,x^2$

$y'' = 2A'(x)\,x + 2A(x) + 3B'(x)\,x^2 + 3B(x)\,(2x)$

i.e., $y'' = 2A'(x)\,x + 2A(x) + 3B'(x)\,x^2 + 6B(x)\,x$

We can apply our second restriction again.

Recall: $A'(x)x^2 + B'(x)x^3 = 0 \Rightarrow A'(x) = -B'(x)x$

Thus, $y'' = 2A'(x)x + 2A(x) + 3B'(x)x^2 + 6B(x)x$

Becomes: $y'' = 2(-B'(x)x)x + 2A(x) + 3B'(x)x^2 + 6B(x)x$

$\Rightarrow y'' = -2B'(x)x^2 + 2A(x) + 3B'(x)x^2 + 6B(x)x$

$\Rightarrow y'' = 2A(x) + B'(x)x^2 + 6B(x)x$

Next, we plug these definitions of y, y', y'' into the original equation: $x^2y'' - 4xy' + 6y = \ln(x)$

$$\begin{array}{rcrlcccc} x^2y'' & = & & 2A(x)x^2 & + & B'(x)x^4 & + & 6B(x)x^3 \\ -4xy' & = & - & 8A(x)x^2 & & & - & 12B(x)x^3 \\ +6y & = & + & 6A(x)x^2 & & & + & 6B(x)x^3 \\ \hline x^2y'' - 4xy' + 6y & = & & & & B'(x)x^4 & & & = & \ln(x) \end{array}$$

$\Rightarrow B'(x)x^4 = \ln(x)$

$\Rightarrow B'(x) = x^{-4}\ln(x)$

We can integrate using Integration by Parts.

$\Rightarrow B(x) = \int B'(x)\,dx = \int \underbrace{\ln(x)}_{u}\underbrace{x^{-4}}_{dv}dx = \int u\,dv = uv - \int v\,du$

$u = \ln(x)$	$dv = x^{-4}$
$\frac{du}{dx} = \frac{1}{x}$	$v = \int x^{-4}dx$
$du = \frac{1}{x}dx$	$v = -\frac{1}{3}x^{-3}$

$= \ln(x)\left(-\frac{1}{3}x^{-3}\right) - \int\left(-\frac{1}{3}x^{-3}\right)\left(\frac{1}{x}\right)dx = -\frac{\ln(x)}{3x^3} + \frac{1}{3}\int x^{-4}dx = -\frac{\ln(x)}{3x^3} + \frac{1}{3}\frac{x^{-3}}{(-3)} + C$

$= -\frac{\ln(x)}{3x^3} - \frac{1}{9x^3} + C$

i.e., $B(x) = -\frac{\ln(x)}{3x^3} - \frac{1}{9x^3} + C_1$

We can apply our second restriction again.

Recall: Our second restriction: $A'(x)\,x^2 + B'(x)\,x^3 = 0 \Rightarrow A'(x) = -B'(x)\,x$

$\Rightarrow A(x) = \int(-B'(x))\,x\,dx = \int(-x^{-4}\ln(x))\,x\,dx = -\int \ln(x)\,x^{-3}\,dx$

$= -\int u\,dv = -\left(uv - \int v\,du\right)$

$u = \ln(x)$	$dv = x^{-3}$
$\frac{du}{dx} = \frac{1}{x}$	$v = \int x^{-3}dx$
$du = \frac{1}{x}dx$	$v = -\frac{1}{2}x^{-2}$

$= -\left(\ln(x)\left(-\frac{1}{2}x^{-2}\right) - \int\left(-\frac{1}{2}x^{-2}\right)\left(\frac{1}{x}dx\right)\right) = \frac{\ln(x)}{2x^2} - \frac{1}{2}\int x^{-3}dx = \frac{\ln(x)}{2x^2} - \frac{1}{2}\frac{x^{-2}}{-2} + C_2$

$= \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + C_2$

i.e., $A(x) = \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + C_2$

Finally, recall: The general solution of our equation is:

$y = A(x)\,x^2 + B(x)\,x^3 = \left(\frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + C_2\right)x^2 + \left(-\frac{\ln(x)}{3x^3} - \frac{1}{9x^3} + C_1\right)x^3$

$= \frac{1}{2}\ln(x) + \frac{1}{4} + C_2x^2 - \frac{1}{3}\ln(x) - \frac{1}{9} + C_1x^3 = \frac{1}{6}\ln(x) + \frac{5}{36} + C_2x^2 + C_1x^3$

$$\boxed{y = \frac{1}{6}\ln(x) + \frac{5}{36} + C_2x^2 + C_1x^3}$$