MTH 3311 – Test #2 - Part #2 - Solutions

Spring 2020

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Directions: Show CLEARLY how you arrive at your answers.

1. The force of water resistance acting on a boat is proportional to its instantaneous velocity, and is such that at 30 $\frac{\text{ft}}{\text{sec}}$ the water resistance is 60 lb. If the boat and passenger combined weigh 480 lb, and if the motor exerts a steady force of 90 lb in the direction of the motion:

   (a) Find the velocity at any time $t \geq 0$, assuming that the boat starts from rest.

   (b) Find the limiting velocity

First, we establish our conventions regarding direction:

Positive Direction $\rightarrow$

We will start our stopwatch at t = 0 sec., and assume that $v(0 \text{ sec}) = \frac{0\,\text{ft}}{\text{sec}}$ (Because "the boat starts from rest")

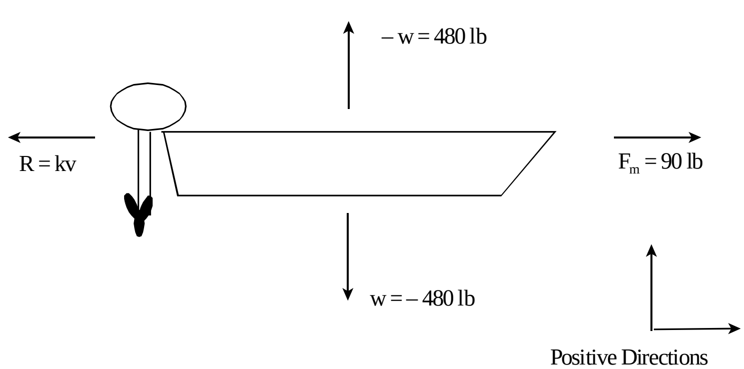
Let R be the water resistance.

LetF_m = Force exerted by the motor

Let w be the combined weight of the boat and the passenger.

Observe that the water exerts a buoyant force equal to $-w$ on the boat (Otherwise, the boat would sink!)

We draw a force diagram on the boat



Also: "water resistance . . . is proportional to its instantaneous velocity" i.e. $R \propto v$

$\Rightarrow R = kv$; where k is the constant of proportionality.

For "future reference," we will find the constant of proportionality right now.

Recall: When $v = 30\frac{\text{ft}}{\text{sec}}$; $R = -60\,\text{lb}$ (Because "at a velocity of $30\frac{\text{ft}}{\text{sec}}$; the water resistance is $60\,\text{lb}$")

Also: $R = kv$

$\Rightarrow -60\,\text{lb} = k\left(30\frac{\text{ft}}{\text{sec}}\right)$

$\Rightarrow k = \frac{-60\,\text{lb}}{30\frac{\text{ft}}{\text{sec}}} = -2\frac{\text{lb sec}}{\text{ft}}$

i.e., $k = -2\frac{\text{lb sec}}{\text{ft}}$ (for "future reference")

Next: We analyze the forces acting on the boat. From the Force Diagram:

1) The sum of the vertical forces is 0 lb.

2) The sum of the horizontal forces is $R + F_m$

Letting F be the total force acting on the boat, we have: $F = R + F_m$

Remark: To allow ourselves to model this relationship as a differential equation, we will employ a **standard trick:**

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Note well: We set F (the sum or all forces on the object) equal to ma

$$\underbrace{\text{Sum of all forces}}_{F} = \underbrace{ma}_{F}$$

This is a standard approach for velocity exercises.

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Our "Standard Trick" yields the equation $\underbrace{R + F_m}_{\text{Sum of all Forces}} = \underbrace{ma}_{F}$ (Eq. 1)

Recall: acceleration is the derivative of velocity. (i.e., $a = \frac{dv}{dt}$)

Thus, Eq. 1 becomes:

$\underbrace{kv + 90\,\text{lb}}_{R+F_m} = m\frac{dv}{dt}$

This is a differential equation in v that we can solve.

$-m\frac{dv}{dt} + kv = -90\,\text{lb}$

$\Rightarrow \frac{dv}{dt} + \underbrace{\left(-\frac{k}{m}\right)}_{P(t)} v = \underbrace{\frac{90\,\text{lb}}{m}}_{Q(t)}$

Our "integrating factor" is $e^{\int P(t)dt} = e^{\int\left(-\frac{k}{m}\right)dt} = e^{-\frac{k}{m}t}$

Multiplying both sides by the integrating factor, we have:

$e^{-\frac{k}{m}t}\frac{dv}{dt} + \left(-\frac{k}{m}\right)e^{-\frac{k}{m}t}v = \frac{90\,\text{lb}}{m}e^{-\frac{k}{m}t}$

$\Rightarrow \frac{d}{dt}\left[e^{-\frac{k}{m}t}v\right] = \frac{90\,\text{lb}}{m}e^{-\frac{k}{m}t}$

$\Rightarrow \int \frac{d}{dt}\left[e^{-\frac{k}{m}t}v\right]dt = \int \frac{90\,\text{lb}}{m}e^{-\frac{k}{m}t}dt$

$\Rightarrow e^{-\frac{k}{m}t}v = \frac{90\,\text{lb}}{m}\left(-\frac{m}{k}e^{-\frac{k}{m}t}\right) + C = -\frac{90\,\text{lb}}{k}e^{-\frac{k}{m}t} + C$

i.e. $e^{-\frac{k}{m}t}v = -\frac{90\,\text{lb}}{k}e^{-\frac{k}{m}t} + C$

$\Rightarrow v = -\frac{90\,\text{lb}}{k} + e^{\frac{k}{m}t}C$

i.e., $v = -\frac{90\,\text{lb}}{k} + Ce^{\frac{k}{m}t}$

$\Rightarrow$**Recall:** $v\,(0\,\text{sec}) = 0\frac{\text{ft}}{\text{sec}}$ (Because "the boat starts from rest.")

$\Rightarrow 0\frac{\text{ft}}{\text{sec}} = v\,(0\,\text{sec}) = -\frac{90\,\text{lb}}{k} + Ce^{\frac{k}{m}(0\text{ sec})} = -\frac{90\,\text{lb}}{k} + C$

$\Rightarrow$ i.e. $0\frac{\text{ft}}{\text{sec}} = -\frac{90\,\text{lb}}{k} + C$

$\Rightarrow$ i.e. $C = \frac{90\,\text{lb}}{k}$

$\Rightarrow v = -\frac{90\,\text{lb}}{k} + \frac{90\,\text{lb}}{k}e^{\frac{k}{m}t}$

Recall Also: $k = -2\frac{\text{lb sec}}{\text{ft}}$

Thus, $v = -\frac{90\,\text{lb}}{\left(-2\frac{\text{lb sec}}{\text{ft}}\right)} + \frac{90\,\text{lb}}{\left(-2\frac{\text{lb sec}}{\text{ft}}\right)}e^{\frac{\left(-2\frac{\text{lb sec}}{\text{ft}}\right)}{m}t} = 45\frac{\text{ft}}{\text{sec}} - 45\frac{\text{ft}}{\text{sec}}e^{\frac{\left(-2\frac{\text{lb sec}}{\text{ft}}\right)}{m}t}$

i.e., $v = 45\frac{\text{ft}}{\text{sec}} - 45\frac{\text{ft}}{\text{sec}}e^{\frac{\left(-2\frac{\text{lb sec}}{\text{ft}}\right)}{m}t}$

All we need to do now is find the value of m.

Recall: $w = mg$ (Where w is the weight of the object, and g is the acceleration due to gravity. i.e., $g = -\frac{32\,\text{ft}}{\text{sec}^2}$)

$\Rightarrow m = \frac{w}{g} = \frac{-480\,\text{lb}}{\left(-\frac{32\,\text{ft}}{\text{sec}^2}\right)} = 15\frac{\text{lb sec}^2}{\text{ft}}$

i.e., $m = 15\frac{\text{lb sec}^2}{\text{ft}}$

Hence, $v = 45\frac{\text{ft}}{\text{sec}} - 45\frac{\text{ft}}{\text{sec}} e^{\frac{\left(-2\frac{\text{lb sec}}{\text{ft}}\right)}{\left(15\frac{\text{lb sec}^2}{\text{ft}}\right)}t} = 45\frac{\text{ft}}{\text{sec}} - 45\frac{\text{ft}}{\text{sec}} e^{-\frac{2}{15\text{ sec}}t}$

$$\boxed{\text{i.e., } v = 45\frac{\text{ft}}{\text{sec}} - 45\frac{\text{ft}}{\text{sec}} e^{-\frac{2}{15\text{ sec}}t}}$$

To find the "limiting velocity," we let $t \to \infty$

$\lim_{t\to\infty} v = \lim_{t\to\infty} \left(45\frac{\text{ft}}{\text{sec}} - 45\frac{\text{ft}}{\text{sec}} e^{-\frac{2}{15\text{ sec}}t}\right) = 45\frac{\text{ft}}{\text{sec}}$

$$\boxed{\text{i.e., Limiting Velocity} = 45\frac{\text{ft}}{\text{sec}}}$$