MTH 1125 - Test 2 (2pm Class) - Solutions

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\frac{d}{dx}\left[3x^6+4x^4+5x^3+7x^2+12x+20\sqrt{x}+10\right]=$

$\frac{d}{dx}\left[3x^6+4x^4+5x^3+7x^2+12x+20x^{\frac{1}{2}}+10\right]$

$=3\left[6x^5\right]+4\left[4x^3\right]+5\left[3x^2\right]+7\left[2x\right]+12+20\left[\frac{1}{2}x^{-\frac{1}{2}}\right]+0$

$=18x^5+16x^3+15x^2+14x+12+10x^{-\frac{1}{2}}$

i.e., $\frac{d}{dx}\left[3x^6+4x^4+5x^3+7x^2+12x+20\sqrt{x}+10\right]=18x^5+16x^3+15x^2+14x+12+10x^{-\frac{1}{2}}$

2. Compute: $\frac{d}{dx}\left[(\sin(x)+\cos(x))(3x^2+6x)\right]=$

$$\frac{d}{dx}\left[\underbrace{(\sin(x)+\cos(x))}_{1^{st}}\underbrace{(3x^2+6x)}_{2^{nd}}\right]=\underbrace{(\cos(x)-\sin(x))}_{1^{\text{st}}\text{ prime}}\cdot\underbrace{(3x^2+6x)}_{2^{nd}}+\underbrace{(6x+6)}_{2^{\text{nd}}\text{ prime}}\cdot\underbrace{(\sin(x)+\cos(x))}_{1^{st}}$$

$\frac{d}{dx}\left[(\sin(x)+\cos(x))(3x^2+6x)\right]=(\cos(x)-\sin(x))(3x^2+6x)+(6x+6)(\sin(x)+\cos(x))$

3. Compute: $\frac{d}{dx}\left[\frac{5x^4+6x^3+16x}{3x^2+6x}\right]=$

$$\frac{d}{dx}\left[\frac{\overbrace{5x^4+6x^3+16x}^{\text{top}}}{\underbrace{3x^2+6x}_{\text{Bottom}}}\right]=\frac{\overbrace{(20x^3+18x^2+16)}^{\text{top prime}}\cdot\overbrace{(3x^2+6x)}^{\text{bottom}}-\overbrace{(6x+6)}^{\text{bottom prime}}\cdot\overbrace{(5x^4+6x^3+16x)}^{\text{top}}}{\underbrace{(3x^2+6x)^2}_{\text{bottom squared}}}$$

i.e., $\frac{d}{dx}\left[\frac{5x^4+6x^3+16x}{3x^2+6x}\right]=\frac{(20x^3+18x^2+16)(3x^2+6x)-(6x+6)(5x^4+6x^3+16x)}{(3x^2+6x)^2}$

4. Compute: $\frac{d}{dx}\left[\left(2x^3+3x^2+6x\right)^8\right] =$ This is the derivative of a function raised to a power.

$$\frac{d}{dx}\left[\left(2x^3+3x^2+6x\right)^8\right] = \underbrace{8\left(2x^3+3x^2+6x\right)^7}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(6x^2+6x+6\right)}_{\substack{\text{derivative}\\ \text{of inner}}}$$

i.e., $\frac{d}{dx}\left[\left(2x^3+3x^2+6x\right)^8\right] = 8\left(2x^3+3x^2+6x\right)^7\left(6x^2+6x+6\right)$

5. Given that $f(x) = 3x^2+3x-3$, give the *equation* of the line tangent to the graph of $f(x)$ at the point $(1,3)$.

We need two things:

   i. A **point** on the line (We have that: $(x_1, y_1) = (1,3)$)

   ii. The **slope** of the line (This is $f'(x_1)$)

   $f'(x) = 6x+3$

At the point $(x_1, y_1) = (1,3)$, **the slope is** $f'(1) = 6(1)+3 = 9$

We will use the Point-Slope equation of a line:

$(y - y_1) = m(x - x_1)$ (Where m is the slope and (x_1, y_1) is a known point on the line.)

Thus, the equation of the line tangent to the graph of $f(x)$ is:

$(y-3) = 9(x-1)$

The equation of the line tangent is $(y-3) = 9(x-1)$

6. Given that $w = \cot(z)$ and that $z = 3x^2 + 6x + 5$; compute $\frac{dw}{dx}$ **using the Liebniz form of the Chain Rule.** (In particular, when doing this exercise, *write explicitly the Liebniz form of the chain rule that you are going to use.)*

We know:

$\frac{dw}{dz} = -\csc^2(z)$

$\frac{dz}{dx} = 6x + 6$

We want: $\frac{dw}{dv}$

By the Liebniz form of the Chain Rule:

$$\frac{dw}{dx} = \frac{dw}{dz}\frac{dz}{dx} = -\csc^2(z)(6x+6) = \underbrace{-\csc^2\left(3x^2+6x+5\right)(6x+6)}_{\substack{\text{express solely in terms of} \\ \text{independent variable } x}}$$

$$\boxed{\text{i.e. } \frac{dw}{dx} = -\csc^2\left(3x^2+6x+5\right)(6x+6)}$$

7. Compute: $\frac{d}{dx}\left[\sin\left(2x^5+5x^2\right)\right] =$

Outer:	=	sin ( )
Deriv. of outer	=	cos ( )

$$\frac{d}{dx}\left[\begin{array}{c} \sin\left(\underbrace{2x^5+5x^2}\right) \\ \uparrow \qquad\quad \uparrow \\ \text{outer} \quad \text{inner} \end{array}\right] = \underbrace{\cos\left(2x^5+5x^2\right)}_{\substack{\text{Deriv of outer,} \\ \text{eval. at inner}}} \cdot \underbrace{\left(10x^4+10x\right)}_{\substack{\text{deriv. of} \\ \text{inner}}}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[\sin\left(2x^5+5x^2\right)\right] = \cos\left(2x^5+5x^2\right)\left(10x^4+10x\right)}$$

8. Compute: $\frac{d}{dx}\left[\left(\frac{5x^2+10x+15}{2x^2+6x}\right)^{100}\right] =$ In the broadest sense, this is the derivative of a *function raised to a power* - USE the CHAIN RULE.

$$\frac{d}{dx}\left[\underbrace{\left(\frac{5x^2+10x+15}{2x^2+6x}\right)^{100}}_{(g(x))^n}\right] = \underbrace{100\left(\frac{5x^2+10x+15}{2x^2+6x}\right)^{99}}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\frac{5x^2+10x+15}{2x^2+6x}\right]\right)}_{\substack{\text{deriv of}\\ \text{inner Function}}}$$

$$= 100\left(\frac{5x^2+10x+15}{2x^2+6x}\right)^{99} \underbrace{\frac{(10x+10)\left(2x^2+6x\right) - (4x+6)\left(5x^2+10x+15\right)}{\left(2x^2+6x\right)^2}}_{\substack{\text{quotient}\\ \text{rule}}}$$

i.e., $\frac{d}{dx}\left[\left(\frac{5x^2+10x+15}{2x^2+6x}\right)^{100}\right] = 100\left(\frac{5x^2+10x+15}{2x^2+6x}\right)^{99} \frac{(10x+10)\left(2x^2+6x\right) - (4x+6)\left(5x^2+10x+15\right)}{\left(2x^2+6x\right)^2}$

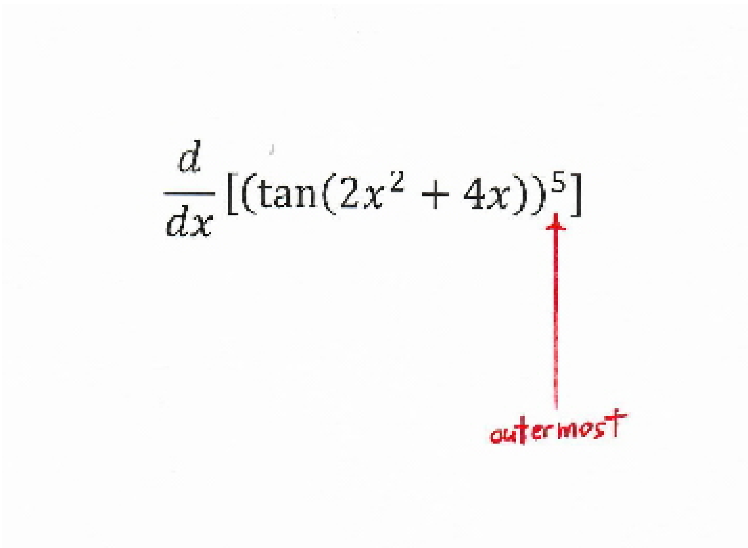
9. Compute: $\frac{d}{dx}\left[\tan^5\left(2x^2+4x\right)\right] =$

Let's rewrite this:

$\frac{d}{dx}\left[\left(\tan\left(2x^2+4x\right)\right)^5\right]$

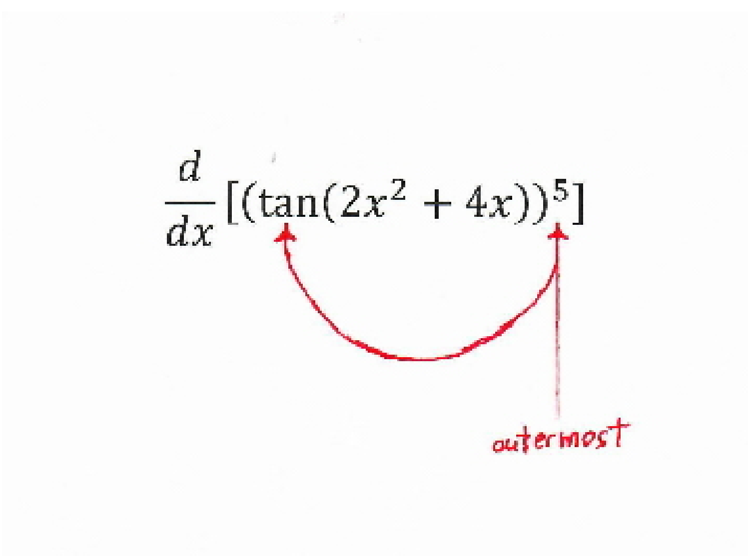
This is the composition of *three* functions.

Differentiate the outermost function and evaluate it at everything inside



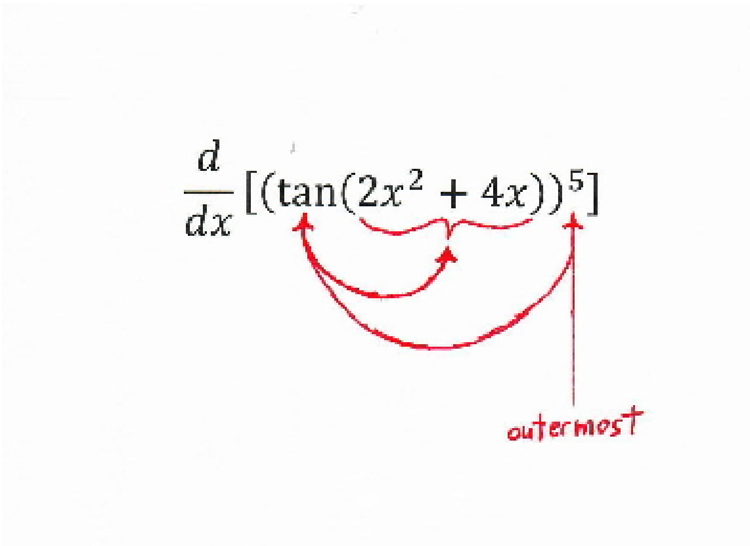
This yields: $5\left(\tan\left(2x^2+4x\right)\right)^4$

Next: Multiply by the derivative of the next outermost function and evaluate it at everything inside of it.



This yields: $5\left(\tan\left(2x^2+4x\right)\right)^4 \cdot \sec^2\left(2x^2+4x\right)$

Finally: Multiply by the derivative of the innermost function.



This yields: $5\left(\tan\left(2x^2+4x\right)\right)^4 \cdot \sec^2\left(2x^2+4x\right) \cdot (4x+4)$

i.e., $\frac{d}{dx}\left[\tan^5\left(2x^2+4x\right)\right] = 5\left(\tan\left(2x^2+4x\right)\right)^4 \cdot \sec^2\left(2x^2+4x\right) \cdot (4x+4)$

Alternatively:

Re-Write!

$\frac{d}{dx}\left[\tan^5\left(2x^2+4x\right)\right] = \frac{d}{dx}\left[\left(\tan\left(2x^2+4x\right)\right)^5\right]$

In the broadest sense, this is *the derivative of a function raised to a power*

$$\frac{d}{dx}\left[\left(\tan\left(2x^2+4x\right)\right)^5\right] = \underbrace{5\left(\tan\left(2x^2+4x\right)\right)^4}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\tan\left(2x^2+4x\right)\right]\right)}_{\substack{\text{derivative}\\ \text{of inner}}}$$

$$= 5\left(\tan\left(2x^2+4x\right)\right)^4 \cdot \underbrace{\left[\sec^2\left(2x^2+4x\right) \cdot (4x+4)\right]}_{\substack{\text{Chain}\\ \text{Rule}}}$$

i.e., $\frac{d}{dx}\left[\tan^5\left(2x^2+4x\right)\right] = 5\left(\tan\left(2x^2+4x\right)\right)^4 \cdot \sec^2\left(2x^2+4x\right) \cdot (4x+4)$

10. Given that $x^3 - x^2y^4 = \tan(y)$, compute $\frac{dy}{dx}$

i. Differentiate both sides w.r.t. x

$$\frac{d}{dx}\left[x^3 - \underbrace{x^2}_{1^{\text{st}}}\underbrace{y^4}_{2^{\text{nd}}}\right] = \frac{d}{dx}[\tan(y)]$$

$$\Rightarrow 3x^2 - \left(\underbrace{2x}_{1^{\text{st}}\text{ prime}} \cdot \underbrace{y^4}_{2^{\text{nd}}} + \underbrace{4y^3\frac{dy}{dx}}_{2^{\text{nd}}\text{ prime}} \cdot \underbrace{x^2}_{1^{\text{st}}}\right) = \sec^2(y) \cdot \frac{dy}{dx}$$

Simplifying, we have:

$3x^2 - 2xy^4 - 4x^2y^3\frac{dy}{dx} = \sec^2(y)\frac{dy}{dx}$

ii. Solve algebraically for $\frac{dy}{dx}$

a. Get $\frac{dy}{dx}$ terms on left side, all other terms on right side

$\Rightarrow -4x^2y^3\frac{dy}{dx} - \sec^2(y)\frac{dy}{dx} = -3x^2 + 2xy^4$

b. Factor out $\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}\left(-4x^2y^3 - \sec^2(y)\right) = -3x^2 + 2xy^4$

c. Divide both sides by the cofactor of $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-3x^2+2xy^4}{-4x^2y^3-\sec^2(y)} = \frac{3x^2-2xy^4}{4x^2y^3+\sec^2(y)}$

$$\boxed{\frac{dy}{dx} = \frac{-3x^2+2xy^4}{-4x^2y^3-\sec^2(y)} = \frac{3x^2-2xy^4}{4x^2y^3+\sec^2(y)}}$$

11. Given that $f(x) = 3x^2 - 4x + 5$, compute $f'(x)$ **using the definition of derivative.** (i.e., using the "limit process.")

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[3(x+\Delta x)^2 - 4(x+\Delta x) + 5\right] - \left[3x^2 - 4x + 5\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[3\left(x^2 + 2x\Delta x + \Delta x^2\right) - 4(x+\Delta x) + 5\right] - \left[3x^2 - 4x + 5\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[3x^2 + 6x\Delta x + 3\Delta x^2 - 4x - 4\Delta x + 5\right] - \left[3x^2 - 4x + 5\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{6x\Delta x + 3\Delta x^2 - 4\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(6x + 3\Delta x - 4)}{\Delta x} = \lim_{\Delta x \to 0} (6x + 3\Delta x - 4) = 6x + 3(0) - 4 = 6x - 4$$

$$\boxed{\text{i.e., } f'(x) = 6x - 4}$$

Extra (Wow! 10 Points)

Given that $T'(x) = \frac{1}{1+x^2}$ (i.e., $\frac{d}{dx}[T(x)] = \frac{1}{1+x^2}$); compute $\frac{d}{dx}[T(\tan(x))]$

$$\boxed{\begin{array}{rcl} \text{Outer:} & = & T(\quad) \\ \text{Deriv. of outer} & = & \frac{1}{1+(\quad)^2} \end{array}}$$

$$\frac{d}{dx}\Big[\underset{\substack{\uparrow \\ \text{outer}}}{T}\Big(\underset{\substack{\uparrow \\ \text{inner}}}{\underbrace{\tan(x)}}\Big)\Big] = \underbrace{\frac{1}{1+(\tan(x))^2}}_{\substack{\text{Deriv of outer,} \\ \text{eval. at inner}}} \cdot \underbrace{\sec^2(x)}_{\substack{\text{deriv. of} \\ \text{inner}}} = \frac{\sec^2(x)}{1+(\tan(x))^2} = \frac{\sec^2(x)}{1+\tan^2(x)} = \frac{\sec^2(x)}{\sec^2(x)} = 1$$

$$\boxed{\text{i.e., } \frac{d}{dx}[T(\tan(x))] = \frac{\sec^2(x)}{1+(\tan(x))^2} = 1}$$