MTH 1126 - Test #1 - Solutions

Spring 2004

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\int \cos(x^2)\, x\, dx =$

   Composite function? Yes! $\cos(x^2)$

   Let u be the "inner function." $\Rightarrow u = x^2$

   Approximate function/derivative pair? Yes! $x^2 \to x$

   Let u be the "function" of the function/derivative pair. $\Rightarrow u = x^2$

   $$\begin{array}{lccl} \text{Let} & u & = & x^2 \\ \Rightarrow & \frac{du}{dx} & = & 2x \\ \Rightarrow & du & = & 2xdx \\ \Rightarrow & \frac{1}{2}du & = & xdx \end{array}$$

   Analyze the integral in terms of u and du.

   $$\int \underbrace{\cos(x^2)}_{\cos(u)} \underbrace{x\, dx}_{\frac{1}{2}du} = \int \cos(u)\, \tfrac{1}{2} du = \tfrac{1}{2} \int \cos(u)\, du$$

   Integrate:

   $$\tfrac{1}{2} \int \cos(u)\, du = \tfrac{1}{2} \sin(u) + C$$

   Re-write in terms of x :

   $$\int \cos(x^2)\, x\, dx = \tfrac{1}{2} \sin(x^2) + C$$

2. Use the "$f - g$" method to compute the area bounded by the graphs of $f(x) = x^2 - 4$ and $g(x) = 2x - 1$.

   First, graph the functions and find the points of intersection.

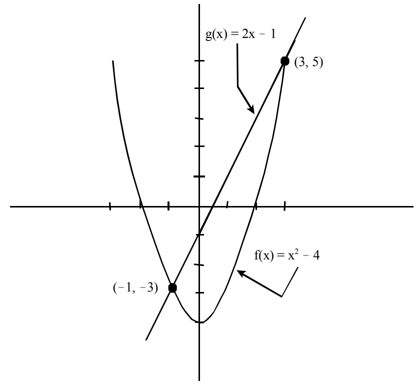
   $y = x^2 - 4 = 2x - 1$

   $\Rightarrow x^2 - 2x - 3 = 0$

   $\Rightarrow (x + 1)(x - 3) = 0$

   $x = -1$; $x = 3$

   Points of intersection are $(-1, -3)$ and $(3, 5)$.



The bounded region spans the interval $[-1,3]$ on the x-axis. Over this interval, $g(x) = 2x-1$ is greater than $f(x) = x^2-4$. Hence the area is given by:

$\int_{-1}^{3}\left[(2x-1)-(x^2-4)\right]dx = \int_{-1}^{3}\left(-x^2+2x+3\right)dx = \left[-\frac{1}{3}x^3+x^2+3x\right]_{-1}^{3}$

$= \left(-\frac{1}{3}(3)^3+(3)^2+3(3)\right)-\left(-\frac{1}{3}(-1)^3+(-1)^2+3(-1)\right) = \frac{32}{3}$

3. Suppose that $\int_2^8 f(x)\,dx = 9$ and that $\int_4^2 f(x)\,dx = 4$. Compute $\int_4^8 f(x)\,dx$.

   Observe: $\int_4^2 f(x)\,dx + \int_2^8 f(x)\,dx = \int_4^8 f(x)\,dx$

   Hence, $\int_4^8 f(x)\,dx = 9+4 = 13$

   Alternatively, $\int_2^4 f(x)\,dx + \int_4^8 f(x)\,dx = \int_2^8 f(x)\,dx$

   Hence, $\int_4^8 f(x)\,dx = \int_2^8 f(x)\,dx - \int_2^4 f(x)\,dx = \int_2^8 f(x)\,dx - \left(-\int_4^2 f(x)\,dx\right) = \int_2^8 f(x)\,dx + \int_4^2 f(x)\,dx = 9+4 = 13$

4. Compute: $\int \left(2x^5-8\right)^{15} x^4\,dx =$

   Composite function? Yes! $\left(2x^5-8\right)^{15}$

   Let u be the "inner function." $\Rightarrow u = 2x^5-8$

   Approximate function/derivative pair? Yes! $(2x^5-8) \to x^4$

   Let u be the "function" of the function/derivative pair. $\Rightarrow u = 2x^5-8$

$$\begin{array}{lll} \text{Let} & u & = 2x^5-8 \\ \Rightarrow & \frac{du}{dx} & = 10x^4 \\ \Rightarrow & du & = 10x^4dx \\ \Rightarrow & \frac{1}{10}du & = x^4dx \end{array}$$

   Analyze in terms of u and du.

$\int \underbrace{\left(2x^5-8\right)^{15}}_{u^{15}} \underbrace{x^4\,dx}_{\frac{1}{10}du} = \int u^{15}\frac{1}{10}du = \frac{1}{10}\int u^{15}du$

Integrate (in terms of u)

$\frac{1}{10}\int u^{15}du = \frac{1}{10}\frac{1}{16}u^{16} + C = \frac{u^{16}}{160} + C$

Re-write in terms of x.

$\int \left(2x^5-8\right)^{15} x^4\,dx = \frac{\left(2x^5-8\right)^{16}}{160} + C$

5. Find the area bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{2}x$. (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

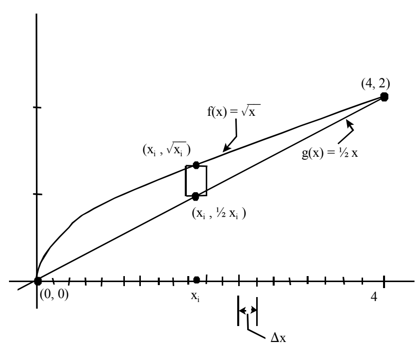
   Graph the functions and find the points of intersection.

   $y = \sqrt{x} = \frac{1}{2}x \Rightarrow x = \frac{1}{4}x^2 \Rightarrow \frac{1}{4}x^2 - x = 0 \Rightarrow x^2 - 4x = 0$

   $\Rightarrow x(x-4) = 0.$

   $\Rightarrow x = 0$; and $x = 4$.

   Points of intersection: $(0,0)$ and $(4,2)$.



   Area of the i^{th} rectangle $= \underbrace{\left(\sqrt{x_i} - \frac{1}{2}x_i\right)}_{\text{height}} \underbrace{\Delta x}_{\text{width}}$

   Total bounded area $\approx \sum_{i=1}^{n}\left(\text{area of } i^{th} \text{ rectangle}\right) = \sum_{i=1}^{n}\left(\sqrt{x_i} - \frac{1}{2}x_i\right)\Delta x$

   Total bounded area $= \lim_{\Delta x \to 0}\sum_{i=1}^{n}\left(\sqrt{x_i} - \frac{1}{2}x_i\right)\Delta x = \int_{x=0}^{x=4}\left(\sqrt{x} - \frac{1}{2}x\right)dx =$

   $\int_{x=0}^{x=4}\left(x^{\frac{1}{2}} - \frac{1}{2}x\right)dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^2\right]_{x=0}^{x=4} = \left(\frac{2}{3}(4)^{\frac{3}{2}} - \frac{1}{4}(4)^2\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}} - \frac{1}{4}(0)^2\right) = \frac{4}{3}.$

   From problems 6 - 8, select two problems.

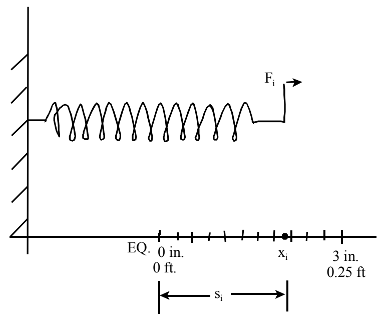
6. 10 pounds of force is required to stretch a spring 6 inches past the point of equilibrium. How much work is done stretching the spring 3 inches past the point of equilibrium? (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

First, find the spring constant, k, using the values $F = 10$ lb and $s = 6$ inches $= \frac{1}{2}$ ft

From Hooke's Law, $F = ks$, we have $k = \frac{F}{s} = \frac{10 \text{ lb}}{\frac{1}{2} \text{ ft}} = 20 \frac{\text{lb}}{\text{ft}}$

Hence, we have: $F = 20 \frac{\text{lb}}{\text{ft}} s$

Next, partition the interval, over which the work is to be performed, and compute W_i, the work done stretching the spring from one side of the i^{th} sub-interval to the other side of the i^{th} sub-interval.



$W_i = F_i d_i$

Here, $d_i = \Delta x$

$F_i = ks_i = 20 \frac{\text{lb}}{\text{ft}} x_i$

Hence, $W_i = 20 \frac{\text{lb}}{\text{ft}} x_i \Delta x$

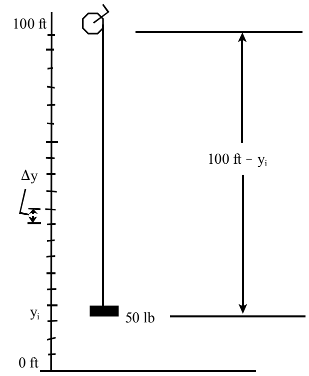
The total work, W_T, is approximately the sum of the work done stretching the spring across each sub-interval.

$W_T \approx \sum_{i=1}^{n} 20 \frac{\text{lb}}{\text{ft}} x_i \Delta x$

$W_T = \lim_{\Delta x \to \infty} \sum_{i=1}^{n} 20 \frac{\text{lb}}{\text{ft}} x_i \Delta x = \int_{0 \text{ ft}}^{0.25 \text{ ft}} 20 \frac{\text{lb}}{\text{ft}} x \, dx = 20 \frac{\text{lb}}{\text{ft}} \int_{0 \text{ ft}}^{0.25 \text{ ft}} x \, dx = 20 \frac{\text{lb}}{\text{ft}} \left[\frac{x^2}{2} \right]_{0 \text{ ft}}^{0.25 \text{ ft}} = 20 \frac{\text{lb}}{\text{ft}} \left[\left(\frac{(0.25 \text{ ft})^2}{2} \right) - \left(\frac{(0 \text{ ft})^2}{2} \right) \right] = \frac{5}{8}$ lb ft

7. A cable, weighing 1 pound per foot length, is used to pull a 50 pound weight from ground level to a height of 100 feet, using a winch. How much work is done in the process? (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

   Partition the interval over which the weight will travel, and compute W_i the work done raising the weight from the bottom to the top of the i^{th} sub-interval.



$W_i = F_i d_i$

Here, $d_i = \Delta y$, and F_i is the combined weight of the unwound portion of the cable plus the 50 lb weight itself.

$F_i = (\text{weight of cable}) + (50 \text{ lb weight})$

$F_i = (\text{length of cable})(\text{weight per unit length}) + (50 \text{ lb})$

$F_i = (100 \text{ ft } - y_i)\left(\frac{1 \text{ lb}}{\text{ft}}\right) + (50 \text{ lb})$

$F_i = \left(100 \text{ lb } - \frac{1 \text{ lb}}{\text{ft}} y_i\right) + (50 \text{ lb})$

$F_i = \left(150 \text{ lb } - \frac{1 \text{ lb}}{\text{ft}} y_i\right)$

Hence, $W_i = F_i d_i = \left(150 \text{ lb } - \frac{1 \text{ lb}}{\text{ft}} y_i\right) \Delta y$

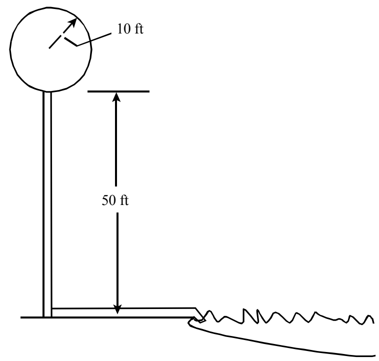
To compute the total work done, W_T, we add up the work done in raising the weight from bottom to top of each sub-interval.

$W_T \approx \sum_{i=1}^{n} \left(150 \text{ lb } - \frac{1 \text{ lb}}{\text{ft}} y_i\right) \Delta y$

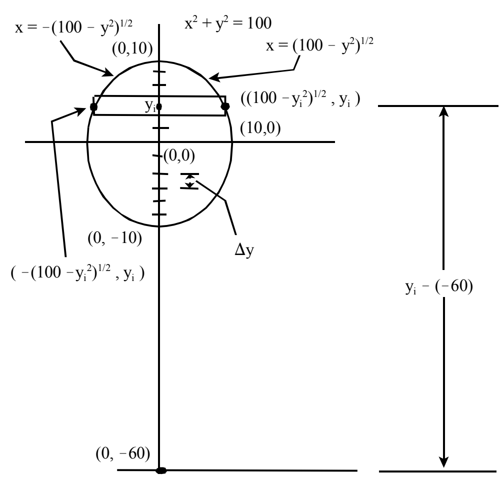
To get the EXACT work done, we let $\Delta y \to 0$.

$W_T = \lim_{\Delta y \to 0} \sum_{i=1}^{n} \left(150 \text{ lb } - \frac{1 \text{ lb}}{\text{ft}} y_i\right) \Delta y = \int_{0 \text{ ft}}^{100 \text{ ft}} \left(150 \text{ lb } - \frac{1 \text{ lb}}{\text{ft}} y\right) dy = \left[150 \text{ lb } y - \frac{1 \text{ lb}}{\text{ft}} \frac{y^2}{2}\right]_{0 \text{ ft}}^{100 \text{ ft}} = \left[\left(150 \text{ lb } (100 \text{ ft}) - \frac{1 \text{ lb}}{\text{ft}} \frac{(100 \text{ ft})^2}{2}\right) - \left(150 \text{ lb } (0 \text{ ft}) - \frac{1 \text{ lb}}{\text{ft}} \frac{(0 \text{ ft})^2}{2}\right)\right] =$ $15,000 \text{ lb ft} - 5,000 \text{ lb ft} = 10,000 \text{ lb ft}$

8. A water tower has a spherical reservior of radius 10 feet. If the bottom of the reservior is 50 feet from ground level, how much work is done filling the reservior by pumping water into the reservior through a hole in the bottom? (Assume that water weighs 100 pounds per cubic foot.) (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)



Situate the tower in the x-y plane. It is probably best to situate the tower so that the center of the sphere coincides with the origin.



Partition the water into horizontal layers, of width Δy.

W_i is the work done pumping the ith layer of water to its final height.

$W_i = F_i d_i$

Here, d_i is the distance that the ith layer of water is pumped.

$d_i = y_i + 60$ ft

F_i is the weight of the ith layer.

$F_i = \big(\text{volume of i}^{th}\text{ layer}\big)\,(\text{weight per unit volume})$

$F_i = (\pi R_i^2 \Delta y)\,\rho$, where $\rho = 100\ \frac{\text{lb}}{\text{ft}^3}$

$$F_i = \left(\pi \left(\sqrt{100\ \text{ft}^2 - y_i^2}\right)^2 \Delta y\right)\rho = \rho\pi\left(100\ \text{ft}^2 - y_i^2\right)\Delta y$$

Hence, $W_i = F_i d_i = \left(\rho\pi\left(100\ \text{ft}^2 - y_i^2\right)\Delta y\right)(y_i + 60\ \text{ft}) = \rho\pi\left(100\ \text{ft}^2 y_i - y_i^3 - 60\ \text{ft}\ y_i^2 + 6{,}000\ \text{ft}\right)\Delta y$

i.e., $W_i = \rho\pi\left(100\ \text{ft}^2 y_i - y_i^3 - 60\ \text{ft}\ y_i^2 + 6{,}000\ \text{ft}\right)\Delta y$

Compute the total work done W_T, by computing the work done pumping each layer to its final height.

$W_T \approx \sum_{i=1}^{n} \rho\pi\left(100\ \text{ft}^2 y_i - y_i^3 - 60\ \text{ft}\ y_i^2 + 6{,}000\ \text{ft}\right)\Delta y$

Let $\Delta y \to 0$

$W_T = \lim_{\Delta y \to 0} \sum_{i=1}^{n} \rho\pi\left(100\ \text{ft}^2 y_i - y_i^3 - 60\ \text{ft}\ y_i^2 + 6{,}000\ \text{ft}\right)\Delta y$

$= \rho\pi \int_{-10\ \text{ft}}^{10\ \text{ft}} \left(100\ \text{ft}^2 y - y^3 - 60\ \text{ft}\ y^2 + 6{,}000\ \text{ft}\right) dy =$

$\rho\pi\left[50\ \text{ft}^2 y^2 - \frac{1}{4}y^4 - 20\ \text{ft}\ y^3 + 6{,}000\ \text{ft}\ y\right]_{-10\ \text{ft}}^{10\ \text{ft}}$

$\rho\pi\left(50\ \text{ft}^2\,(10\ \text{ft})^2 - \frac{1}{4}(10\ \text{ft})^4 - 20\ \text{ft}\ (10\ \text{ft})^3 + 6{,}000\ \text{ft}\ (10\ \text{ft})\right)$
$-\rho\pi\left(50\ \text{ft}^2\,(-10\ \text{ft})^2 - \frac{1}{4}(-10\ \text{ft})^4 - 20\ \text{ft}\ (-10\ \text{ft})^3 + 6{,}000\ \text{ft}\ (-10\ \text{ft})\right) = \rho\pi\left(80{,}000\ \text{ft}^4\right) =$
$100\frac{\text{lb}}{\text{ft}^3}\pi\left(80{,}000\ \text{ft}^4\right) = 8{,}000{,}000\ \pi$ lb ft