# MTH 1126 - Test #2 (11am Class) - Solutions

Spring 2024

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**Instructions.** Show CLEARLY how you arrive at your answers.

1. Use the "$f - g$" method to compute the area bounded by the graphs of $f(x) = -x^2 + 4$ and $g(x) = x + 2$.

   First, graph the functions and find the points of intersection.

   $y = -x^2 + 4 = x + 2$

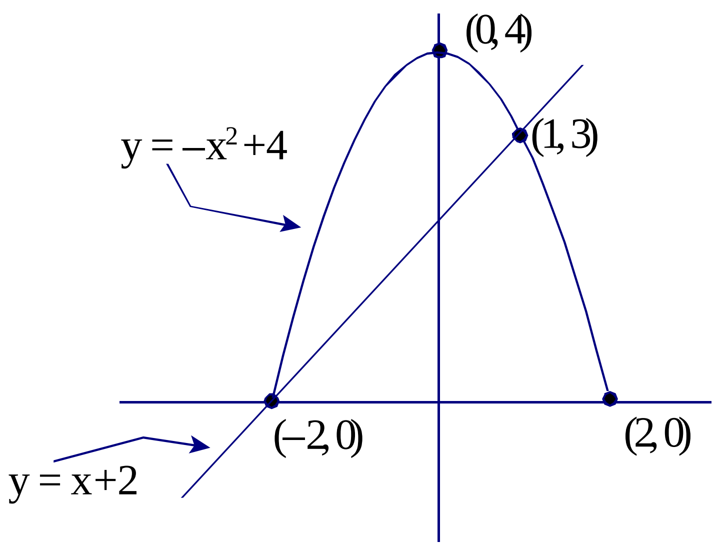
   i.e., $-x^2 - x + 2 = 0$

   $\Rightarrow x^2 + x - 2 = 0$

   $\Rightarrow (x + 2)(x - 1) = 0$

   $x = -2$; $x = 1$

   Points of intersection are $(-2, 0)$ and $(1, 3)$.




   The bounded region spans the interval $[-2, 1]$ on the $x$-axis. Over this interval, $f(x) = -x^2 + 4$ is greater than $g(x) = x + 2$. Hence the area is given by:

   $\int_{-2}^{1} ((-x^2 + 4) - (x + 2))\, dx = \int_{-2}^{1} (-x^2 - x + 2)\, dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$

   $= \left(-\frac{(1)^3}{3} - \frac{(1)^2}{2} + 2(1)\right) - \left(-\frac{(-2)^3}{3} - \frac{(-2)^2}{2} + 2(-2)\right) = \frac{9}{2}$

i.e., bounded area $= \frac{9}{2}$

2. Find the area bounded by the graphs of $f(x) = 4x$ and $g(x) = x^2$. (Partition the appropriate interval, sketch the i$^{\text{th}}$ rectangle, build the Riemann Sum, derive the appropriate integral.)

Graph the functions and find the points of intersection.

To find the points of intersection, set the y-coordinates equal to one another and solve for x.

$y = 4x = x^2$

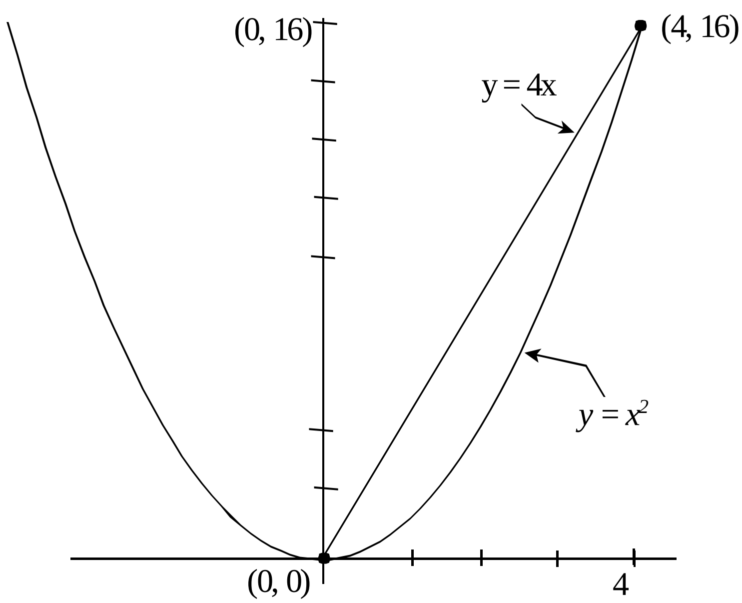
$\Rightarrow 4x - x^2 = 0$

$\Rightarrow x^2 - 4x = 0$

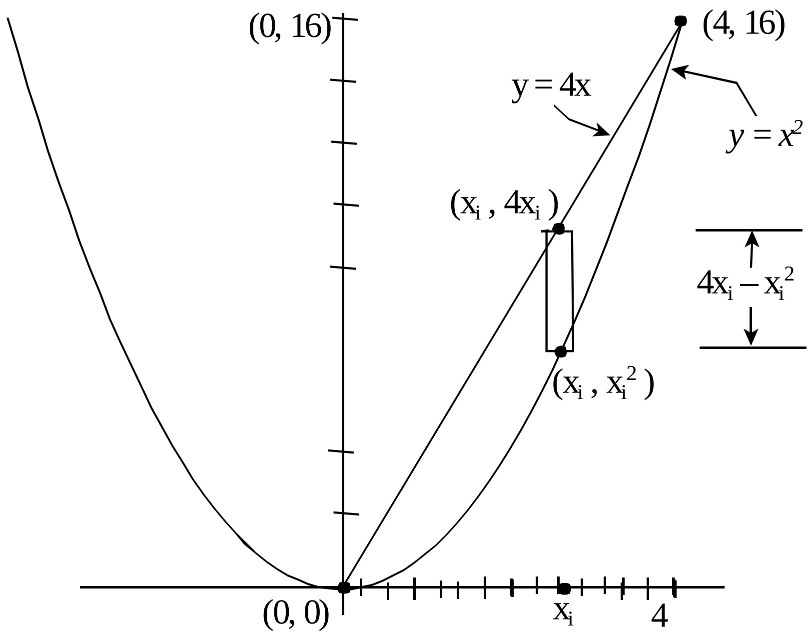
$\Rightarrow x(x-4) = 0$

$\Rightarrow x = 0$; and $x = 4$.

Points of intersection: $(0,0)$ and $(4,16)$.



Inscribe a rectangle of width $\triangle x$ between the two graphs.




The rectangles span the interval $[0, 4]$ on the $x$-axis, so we will partition that interval into sub-intervals of length $\Delta x$.

The area of the $i^{th}$. rectangle is $\underbrace{\left(4x_i - x_i^2\right)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}}$

To approximate the area of the bounded region, we add the areas of the rectangles:

$A \approx \sum_{i=1}^{n} \left(4x_i - x_i^2\right) \Delta x$

To get the exact area, we let $\Delta x \to 0$.

$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \left(4x_i - x_i^2\right) \Delta x = \int_0^4 \left(4x - x^2\right) dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4$

$= \left(2\,(4)^2 - \frac{1}{3}\,(4)^3\right) - \left(2\,(0)^2 - \frac{1}{3}\,(0)^3\right) = \frac{32}{3}$

i.e., bounded area $= \frac{32}{3}$

3. Use the "shell method" to compute the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = \sqrt{x}$, and $y = x^2$, about the $y$-axis. (For your information: the equation of the $y$-axis is $x = 0$.)

Use the "five step method" (partition the interval, sketch the $i^{th}$ rectangle, form the sum, take the limit)

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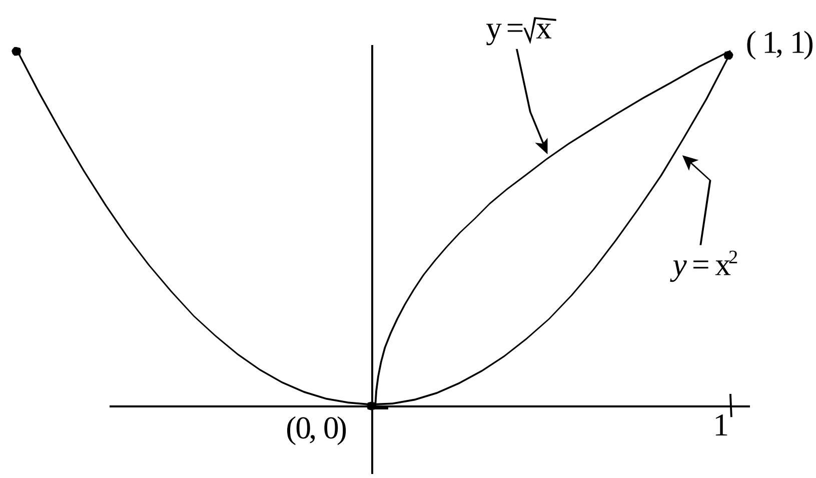
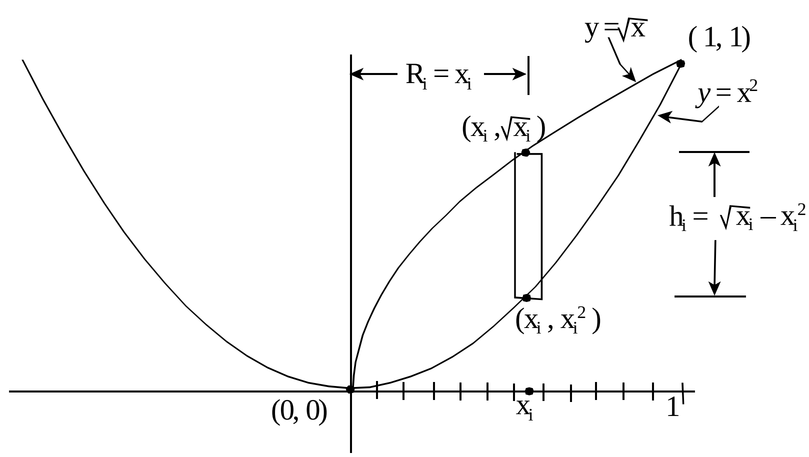
i) First, we'll graph the bounded region.




ii) Next, we sketch a rectangle of width $\Delta x$ parallel ("shell-parallel") to the axis of revolution, and we partition the interval spanned by the rectangles.



iii) Revolve the $i^{th}$ rectangle about the axis of revolution and compute the volume of the $i^{th}$ shell, $Vol_i$

$$Vol_i = 2\pi R_i h_i \Delta x = 2\pi x_i \left(\sqrt{x_i} - x_i^2\right) \Delta x = 2\pi x_i \left(x_i^{\frac{1}{2}} - x_i^2\right) \Delta x$$

$$= 2\pi \left(x_i^{\frac{3}{2}} - x_i^3\right) \Delta x$$

iv) Approximate the volume of the solid by adding up the volumes of the shells

$$Vol \approx \sum_{i=1}^{n} 2\pi \left(x_i^{\frac{3}{2}} - x_i^3\right) \Delta x$$

v) Let $\Delta x \to 0$

$$Vol = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 2\pi \left(x_i^{\frac{3}{2}} - x_i^3\right) \Delta x = \int_{x=0}^{x=1} 2\pi \left(x^{\frac{3}{2}} - x^3\right) dx = 2\pi \left[\frac{5x^{\frac{5}{2}}}{2} - \frac{x^4}{4}\right]_{x=0}^{x=1}$$

$$= 2\pi \left[\left(\frac{5(1)^{\frac{5}{2}}}{2} - \frac{(1)^4}{4}\right) - \left(\frac{5(0)^{\frac{5}{2}}}{2} - \frac{(0)^4}{4}\right)\right] = \frac{9}{2}\pi$$

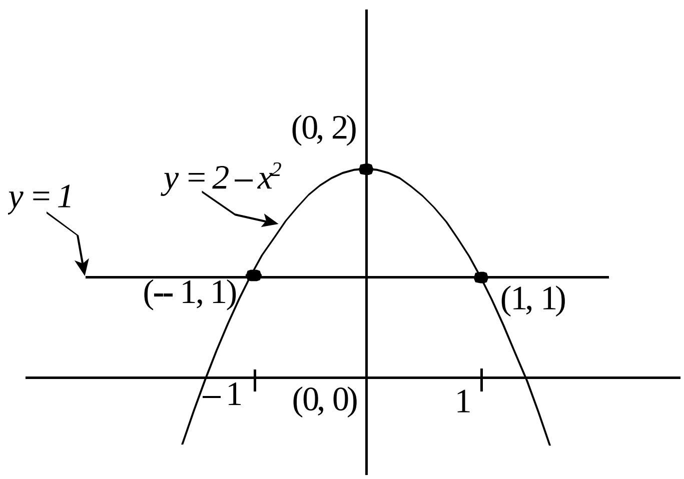
$$\boxed{Vol = \frac{9}{2}\pi}$$

4. Use the "disc method" to compute the volume of the solid of revolution generated by revolving the region described below about the $x$-axis.

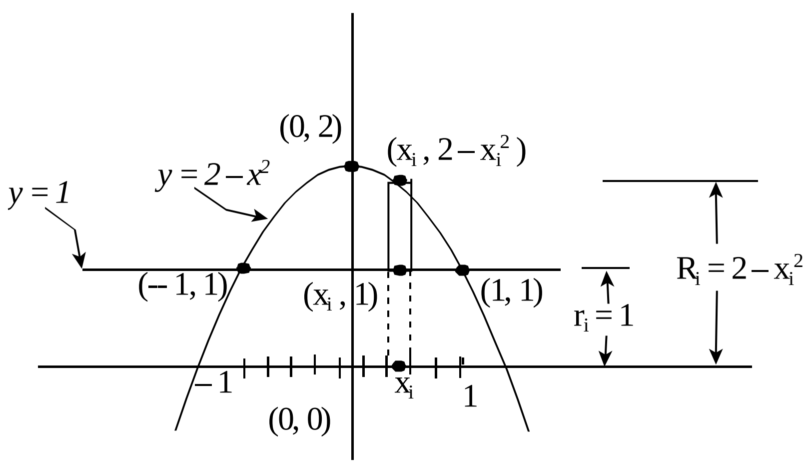
The region lies above the $x$-axis and is bounded by the graphs $y = 2 - x^2$ and $y = 1$.

Use the "five step method" (partition the interval, sketch the i$^{\text{th}}$ rectangle, form the sum, take the limit)

i) First, we'll graph the bounded region.




ii) Next, we sketch a rectangle of width $\Delta x$ perpendicular (perpen-"disc"-ular) to the axis of revolution, and we partition the interval spanned by the rectangles.




iii) Revolve the i$^{\text{th}}$ rectangle about the axis of revolution and compute the volume of the i$^{\text{th}}$ disc (or i$^{\text{th}}$ washer), $Vol_i$

$$Vol_i = \text{Vol i}^{\text{th}} \text{ large disc} - \text{Vol i}^{\text{th}} \text{ hole}$$

$$= \pi R_i^2 \Delta x - \pi r_i^2 \Delta x = \pi \left(2 - x_i^2\right)^2 \Delta x - \pi \left(1\right)^2 \Delta x$$

$$= \pi \left(x_i^4 - 4x_i^2 + 4\right) \Delta x - \pi \left(1\right) \Delta x = \pi \left(x_i^4 - 4x_i^2 + 3\right) \Delta x$$

iv) Approximate the volume of the solid by adding up the volumes of the discs (washers)

$Vol \approx \sum_{i=1}^{n} \pi \left(x_i^4 - 4x_i^2 + 3\right) \Delta x$

v) Let $\Delta x \to 0$

$Vol = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi \left(x_i^4 - 4x_i^2 + 3\right) \Delta x = \int_{x=-1}^{x=1} \pi \left(x^4 - 4x^2 + 3\right) dx$

$= \pi \left[\frac{1}{5}x^5 - \frac{4}{3}x^3 + 3x\right]_{x=-1}^{x=1}$

$= \pi \left(\frac{1}{5}(1)^5 - \frac{4}{3}(1)^3 + 3(1)\right) - \pi \left(\frac{1}{5}(-1)^5 - \frac{4}{3}(-1)^3 + 3(-1)\right)$

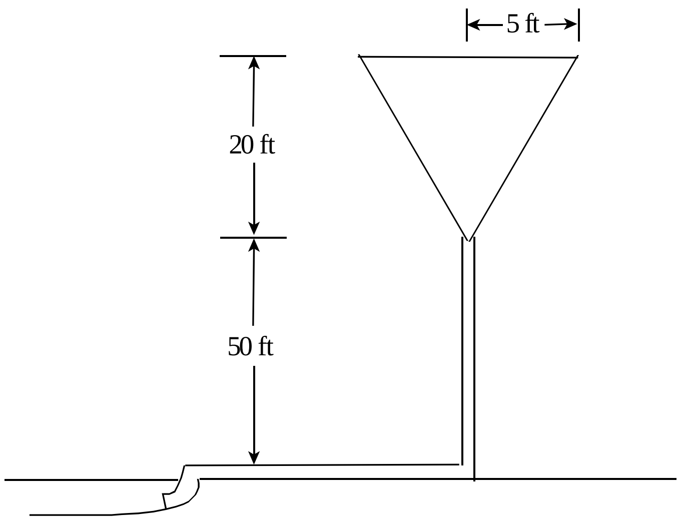
$= \frac{56}{15}\pi$

$$\boxed{Vol = \frac{56}{15}\pi}$$

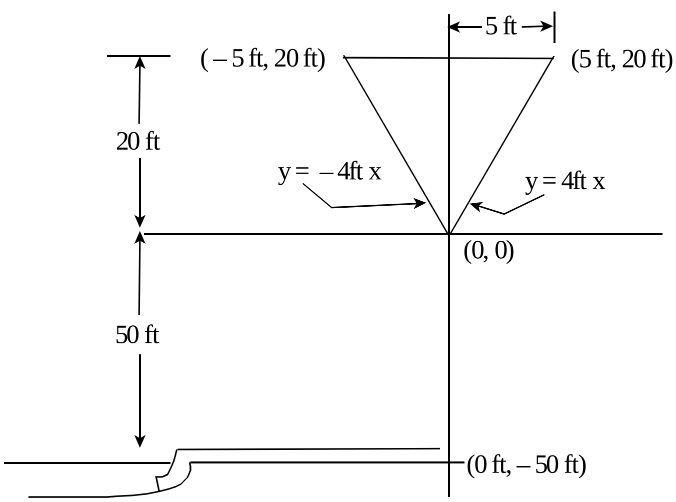
From exercises 5 and 6, select one.

5. Compute the work done in filling the reservoir of a water tower, though a hole in the bottom of the reservoir. The reservoir is a "cone-shaped" tank of height 20 ft and radius 5 ft at the top. The base of the reservoir is 50 ft above the level of the pond from which the water is pumped. (Assume that water weighs $\rho = 100\frac{\text{lbs}}{\text{ft}^3}$)

   The "layout" is depicted below:



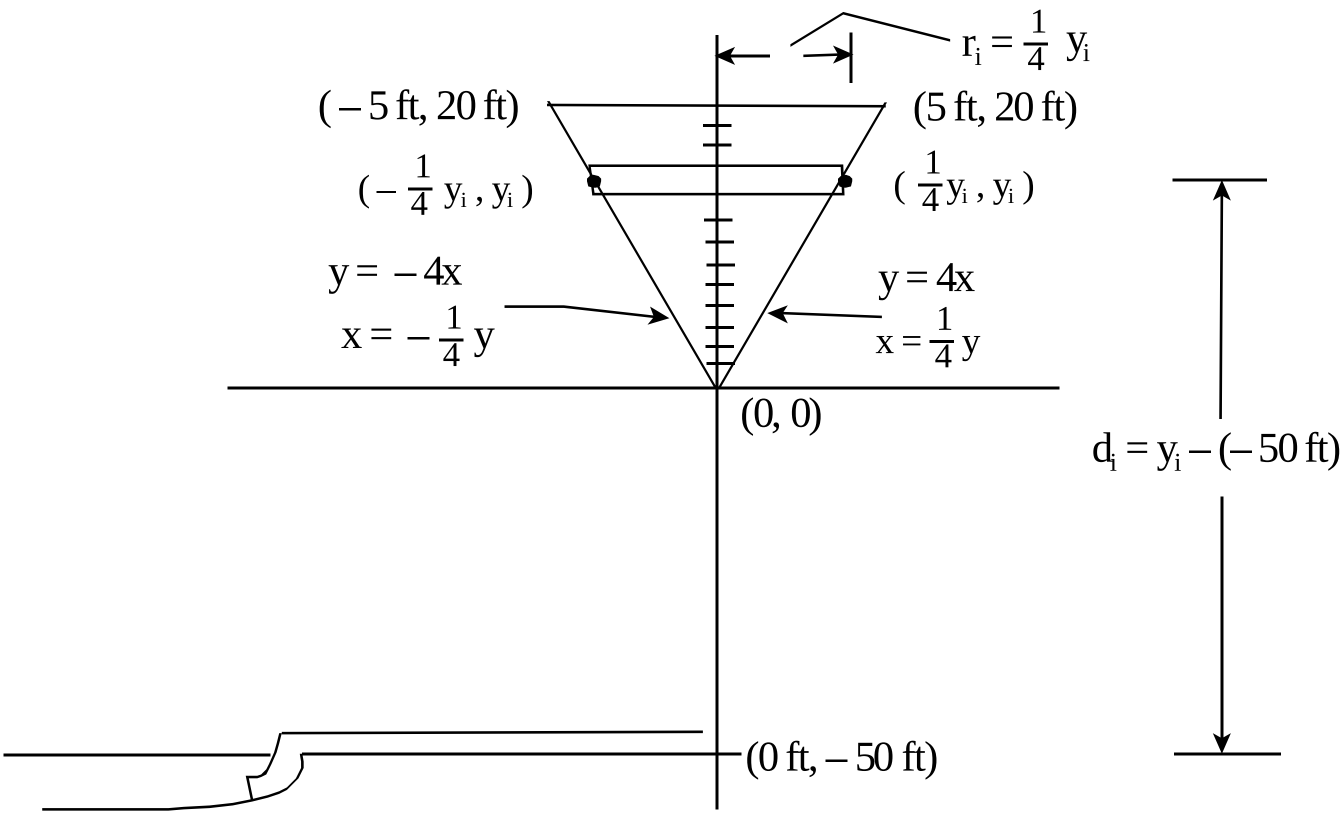

   For convenience, we'll situate the reservoir so that the vertex is at the origin.




   We'll partition the water into "slices" of thickness $\triangle x$ and compute the work done pumping the $i^{th}$ slice from ground level to it's final height.

   Our assumption is that the $i^{th}$ slice is so thin that every molecule within the slice is approximately the same distance from ground level.

   Thus, the distance over which the work in done in pumping the $i^{th}$ slice to its final height will be constant.




We compute $W_i$ the work done pumping the $i^{th}$ slice to its final height.

$W_i = F_i \cdot d_i = w_i \cdot d_i$ where $w_i$ is the weight of the $i^{th}$ slice of water

$$w_i = \left(\text{volume of } i^{th} \text{ slice}\right) \cdot (\text{weight per unit volume})$$

$$= (\text{cross-sectional area } \cdot \text{ thickness}) \cdot \left(\rho = 100\tfrac{\text{lbs}}{\text{ft}^3}\right)$$

$$= (\pi r_i^2 \cdot \triangle y) \cdot (100\tfrac{\text{lbs}}{\text{ft}^3}) = 100\tfrac{\text{lbs}}{\text{ft}^3}\pi r_i^2 \cdot \triangle y = 100\tfrac{\text{lbs}}{\text{ft}^3}\pi\left(\tfrac{1}{4}y_i\right)^2 \cdot \triangle y$$

$$= 100\tfrac{\text{lbs}}{\text{ft}^3}\pi\left(\tfrac{1}{16}y_i^2\right) \cdot \triangle y = \tfrac{25}{4}\tfrac{\text{lbs}}{\text{ft}^3}\pi y_i^2 \cdot \triangle y$$

i.e., $w_i = \frac{25}{4}\frac{\text{lbs}}{\text{ft}^3}\pi y_i^2 \cdot \triangle y$

Thus, $W_i = F_i \cdot d_i = w_i \cdot d_i = \left(\frac{25}{4}\frac{\text{lbs}}{\text{ft}^3}\pi y_i^2 \cdot \triangle y\right)(y_i - (-50\text{ ft}))$

$$= \left(\tfrac{25}{4}\tfrac{\text{lbs}}{\text{ft}^3}\pi y_i^2 \cdot \triangle y\right)(y_i + 50\text{ ft}) = \tfrac{25}{4}\tfrac{\text{lbs}}{\text{ft}^3}\pi y_i^2 (y_i + 50\text{ ft})\triangle y$$

$$= \tfrac{25}{4}\tfrac{\text{lbs}}{\text{ft}^3}\pi\left(y_i^3 + 50\text{ ft}y_i^2\right)\triangle y$$

i.e., $W_i = \frac{25}{4}\frac{\text{lbs}}{\text{ft}^3}\pi\left(y_i^3 + 50\text{ ft}y_i^2\right)\triangle y$

The Total Work Done in filling the reservoir $W_T$ is given by:

$$W_T \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} \frac{25}{4}\frac{1\text{bs}}{\text{ft}^3}\pi\left(y_i^3 + 50\text{ ft}y_i^2\right)\triangle y$$

To get the exact work done, we let $\triangle y \to 0$

$$W_T = \lim_{\triangle y\to 0}\sum_{i=1}^{n} W_i = \lim_{\triangle y\to 0}\sum_{i=1}^{n} \frac{25}{4}\frac{1\text{bs}}{\text{ft}^3}\pi\left(y_i^3 + 50\text{ ft}y_i^2\right)\triangle y$$

$$= \int_{y=0\text{ft}}^{y=20\text{ft}} \frac{25}{4}\frac{1\text{bs}}{\text{ft}^3}\pi\left(y^3 + 50\text{ ft}y^2\right)dy = \frac{25}{4}\frac{1\text{bs}}{\text{ft}^3}\pi\left[\frac{y^4}{4} + 50\text{ ft}\frac{y^3}{3}\right]_{0\text{ft}}^{20\text{ft}}$$

$$= \frac{25}{4}\frac{1\text{bs}}{\text{ft}^3}\pi\left[\frac{(20\text{ft})^4}{4} + 50\text{ ft}\frac{(20\text{ft})^3}{3}\right] - \frac{25}{4}\frac{1\text{bs}}{\text{ft}^3}\pi\left[\frac{(0\text{ft})^4}{4} + 50\text{ ft}\frac{(0\text{ft})^3}{3}\right]$$

$$= \frac{25}{4}\frac{1\text{bs}}{\text{ft}^3}\pi\left[40,000\text{ft}^4 + \frac{400,000}{3}\text{ft}^4\right] = \frac{25}{4}\frac{1\text{bs}}{\text{ft}^3}\pi\left[\frac{520,000}{3}\text{ft}^4\right] = \frac{3,250,000}{3}\pi\text{ lb ft}$$

i.e., $W = \frac{3,250,000}{3}\pi$ lb ft

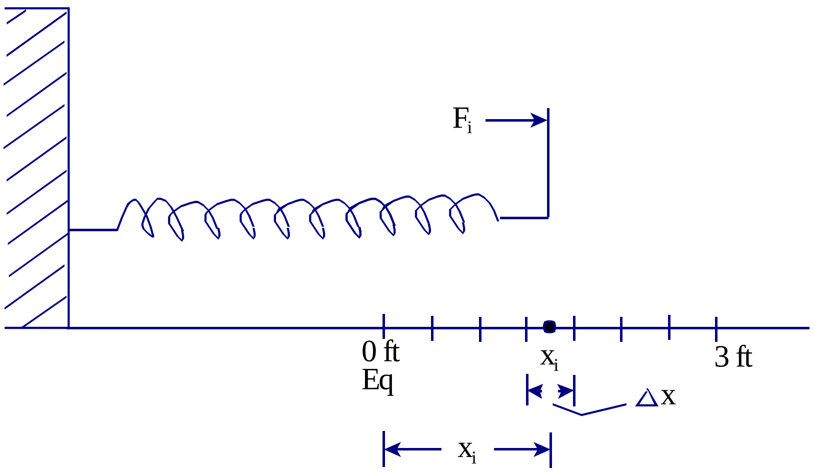
6. 20 lb of force is required to stretch a spring 1ft past the point of equilibrium. Compute the work done in stretching the spring a distance of 3 ft past the point of equilibrium.

First, find the spring constant, $k$, using the values $F = 20$ lb and $x = 1$ ft

From Hooke's Law, $F = kx$, we have $k = \frac{F}{x} = \frac{20 \text{ lb}}{1 \text{ ft}} = 20 \ \frac{\text{lb}}{\text{ft}}$

Hence, we have: $F = 20\frac{\text{lb}}{\text{ft}}x$

Next, partition the interval, over which the work is to be performed, and compute $W_i$, the work done stretching the spring from one side of the $i^{th}$ sub-interval to the other side of the $i^{th}$ sub-interval.




$W_i = F_i d_i$

Here, $d_i = \Delta x$

$F_i = kx_i = 20\frac{\text{lb}}{\text{ft}}x_i$

Hence, $W_i = 20\frac{\text{lb}}{\text{ft}}x_i \triangle x$

The total work, $W_T$, is approximately the sum of the work done stretching the spring across each sub-interval.

$W_T \approx \sum_{i=1}^{n} 20\frac{\text{lb}}{\text{ft}}x_i \triangle x$

$W_T = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 20\frac{\text{lb}}{\text{ft}}x_i \triangle x = \int_{0 \text{ ft}}^{3 \text{ ft}} 20\frac{\text{lb}}{\text{ft}}x \, dx = 20\frac{\text{lb}}{\text{ft}} \int_{0 \text{ ft}}^{3 \text{ ft}} x \, dx = 20\frac{\text{lb}}{\text{ft}} \left[\frac{x^2}{2}\right]_{0 \text{ ft}}^{3 \text{ ft}}$

$= 20\frac{\text{lb}}{\text{ft}} \left[\left(\frac{(3 \text{ ft})^2}{2}\right) - \left(\frac{(0 \text{ ft})^2}{2}\right)\right] = 90 \text{ lb ft}$

$\boxed{W_T = 90 \text{ lb ft}}$