MTH 1125 (12 pm) Test #3 - Solutions

FALL 2019

Pat Rossi Name ______________________

Instructions. Show CLEARLY how you arrive at your answers.

1. $f(x) = x^3 - 12x + 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

   i. Compute $f'(x)$ and find the critical numbers

   $f'(x) = 3x^2 - 12$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 0$ and solve for x

   $\Rightarrow f'(x) = 3x^2 - 12 = 0$

   $\Rightarrow 3x^2 - 12 = 0$

   $\Rightarrow x^2 - 4 = 0$

   $\Rightarrow (x+2)(x-2) = 0$

   $\Rightarrow x = -2$ and $x = 2$ are critical numbers.

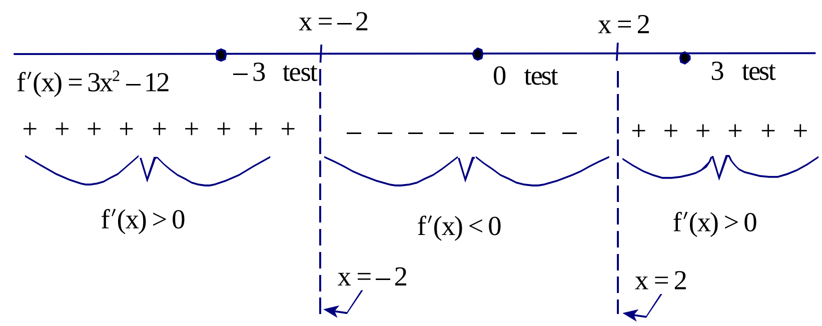
   "Type b" $(f'(c)$ is undefined)

   Look for x-value that causes division by zero.

   No "type b" critical numbers

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

3. Pick a "test point" from each interval to plug into $f'(x)$



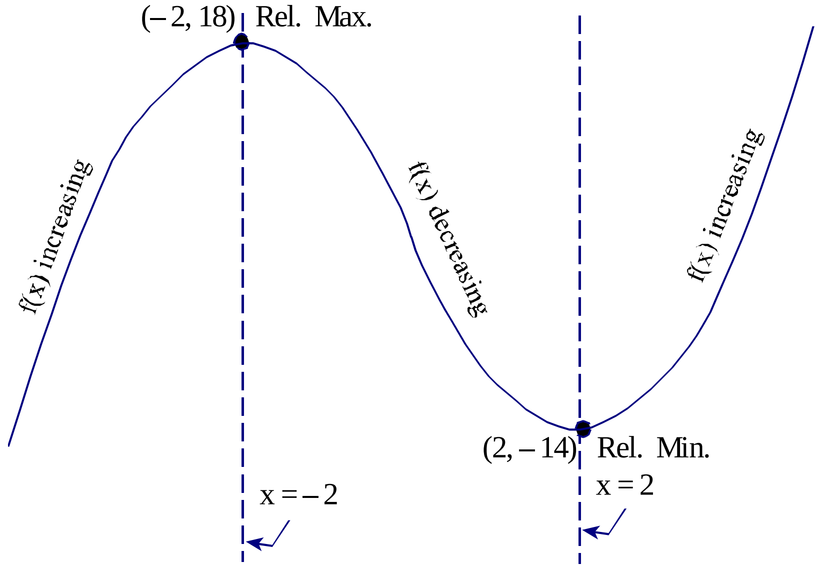
$f(x)$ is **increasing** on the interval(s) $(-\infty, -2)$ and $(2, \infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-2, 2)$

(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(-2, f(-2)) = (-2, 18)$

Rel Min $(2, f(2)) = (2, -14)$

2. $f(x) = x^4 - 2x^3 - 12x^2 + 6x + 3$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection.

   1. Compute $f''(x)$ and find possible points of inflection.

      $f'(x) = x^4 - 2x^3 - 12x^2 + 6x + 3$

      $f''(x) = 12x^3 - 12x - 24$

      Find possible points of inflection:

      a. "Type a" $(f''(x) = 0)$

         Set $f''(x) = 0$

         $\Rightarrow f''(x) = 12x^2 - 12x - 24 = 0$

         $\Rightarrow x^2 - x - 2 = 0$

         $\Rightarrow (x+1)(x-2) = 0$

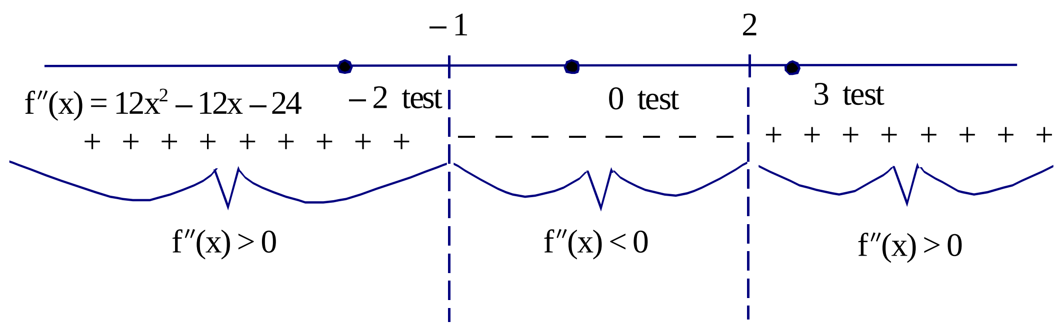
         $x = -1, 2$ possible "type a" points of inflection

      b. "Type b" $(f''(x)$ undefined$)$

         No "Type b" points of inflection

   2. Draw a "sign graph" of $f''(x)$, using the possible points of inflection to partition the x-axis.

   3. Select a test point from each interval and plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-\infty, -1)$ and $(2, \infty)$
(because $f''(x)$ is positive on these intervals)

$f(x)$ is **concave down** on the interval $(-1, 2)$
(because $f''(x)$ is negative on this interval)

Since $f(x)$ changes concavity at $x = -1$ and $x = 2,$ the points:
$(-1, f(-1)) = (-1, -12)$
and
$(2, f(2)) = (2, -33)$ **are** points of inflection.

3. $f(x) = 2x^3 + 9x^2 - 24x + 2$ on the interval $[-2, 2]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

Note: [1] $f(x)$ is continuous (since it is a polynomial) on the [2]closed, [3]finite interval $[-2, 2]$. Therefore, we can use the Absolute Max/Min Value Test.

i. Compute $f'(x)$ and find the critical numbers.

$f'(x) = 6x^2 + 18x - 24$

a. "Type a" ($f'(x) = 0$)

$f'(x) = 6x^2 + 18x - 24 = 0$

$\Rightarrow x^2 + 3x - 4 = 0$

$\Rightarrow (x + 4)(x - 1) = 0$

$\Rightarrow x = -4, 1$ are "type a" critical numbers

Since $x = -4$ is not in the interval $[-2, 2]$, we discard it as a critical number.

b. "Type b" ($f'(x)$ is undefined)

No "Type b" critical numbers

ii. Plug endpoints and critical numbers into $f(x)$ (the *original* function)

$f(-2) = 2(-2)^3 + 9(-2)^2 - 24(-2) + 2 = 70$ $\leftarrow$ Abs Max Value

$f(1) = 2(1)^3 + 9(1)^2 - 24(1) + 2 = -11$ $\leftarrow$ Abs Min Value

$f(2) = 2(2)^3 + 9(2)^2 - 24(2) + 2 = 6$

The Abs Max Value is 70 (attained at $x = -2$) The Abs Min Value is -11 (attained at $x = 1$)

4. $f(x) = x^{\frac{8}{3}} - 4x^{\frac{2}{3}}$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

1. Compute $f'(x)$ and find the critical numbers

   $f'(x) = \frac{8}{3}x^{\frac{5}{3}} - \frac{8}{3}x^{-\frac{1}{3}} = \frac{8x^{\frac{5}{3}}}{3} - \frac{8}{3x^{\frac{1}{3}}} = \frac{8x^{\frac{5}{3}}}{3}\frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}} - \frac{8}{3x^{\frac{1}{3}}} = \frac{8x^2-8}{3x^{\frac{1}{3}}}$

   i.e., $f'(x) = \frac{8x^2-8}{3x^{\frac{1}{3}}}$

   a. "Type a" $(f'(c) = 0)$

      Set $f'(x) = 0$ and solve for x

      $\Rightarrow f'(x) = \frac{8x^2-8}{3x^{\frac{1}{3}}} = 0$

      $\Rightarrow 8x^2 - 8 = 0$

      $\Rightarrow x^2 - 1 = 0$

      $\Rightarrow (x+1)(x-1) = 0$

      $\Rightarrow x = -1$ and $x = 1$ are critical numbers.

   b. "Type b" $(f'(c)$ is undefined)

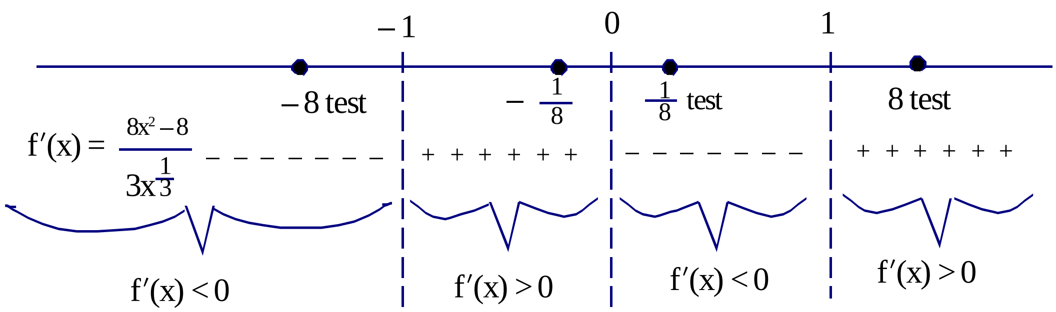
      Look for x-value that causes division by zero.

      $\Rightarrow 3x^{\frac{1}{3}} = 0$

      $\Rightarrow x = 0$ "type b" critical number

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

3. Pick a "test point" from each interval to plug into $f'(x)$



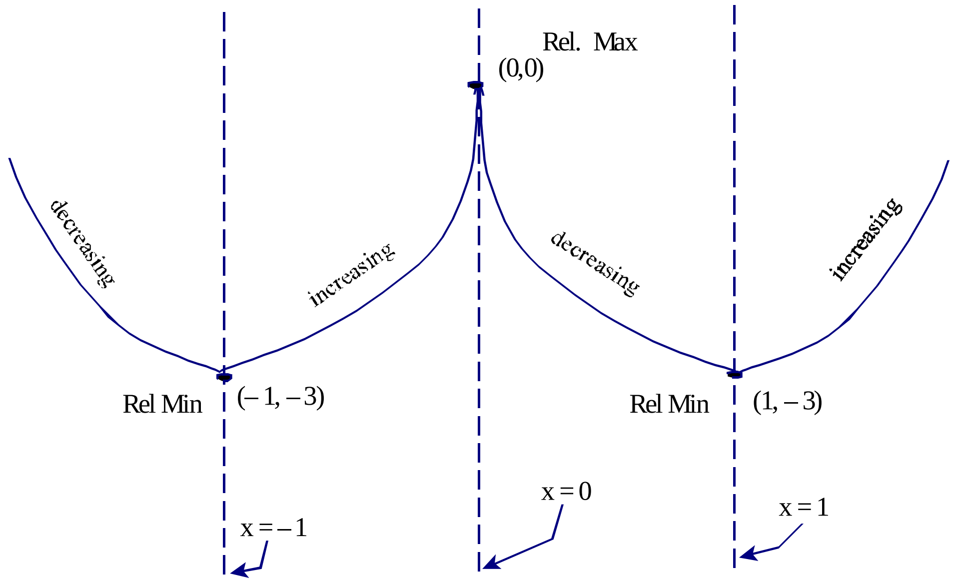
$f(x)$ is **increasing** on the interval(s) $(-1, 0)$ and $(1, \infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-\infty, -1)$ and $(0, 1)$

(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.

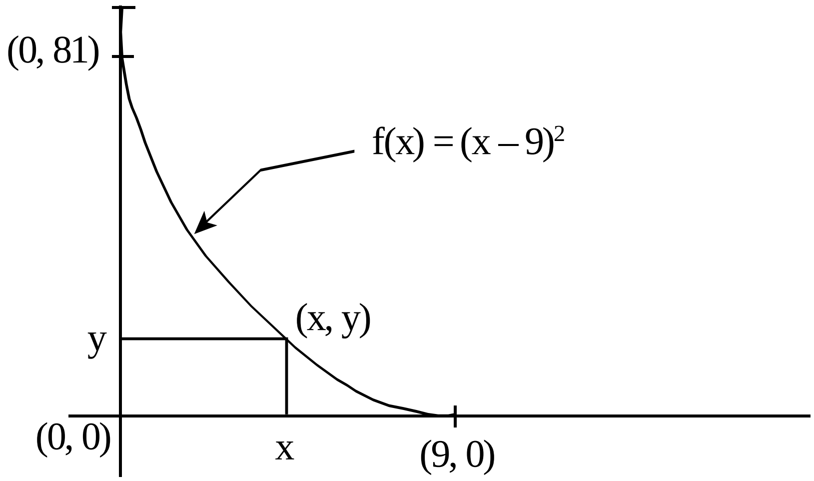


Rel Minimums: $(-1, f(-1)) = (-1, -3)$

and $(1, f(1)) = (1, -3)$

Rel Maximum: $(0, f(0)) = (0, 0)$

5. A rectangle is inscribed in the region bounded by the positive x-axis, the positive y-axis, and the graph of $f(x) = (x-9)^2$ as shown below. Determine the value of x that makes the area of the rectangle as large as possible.



1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = xy$

   a. Draw a picture where relevant.

      (Done)

2. Express A as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that the point (x, y) must be on the graph of $f(x) = (x-9)^2$.

   Hence, the y-coordinate of the point (x, y) is $y = (x-9)^2$.

   Plug this into the equation $A = xy$

   $\Rightarrow A(x) = x(x-9)^2 = x^3 - 18x^2 + 81x$

   i.e., $A(x) = x^3 - 18x^2 + 81x$

3. Determine the restrictions on the independent variable x.

   From the picture, $0 \leq x \leq 9$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0, 9]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = 3x^2 - 36x + 81$

      a. "Type a" $(f'(c) = 0)$

         $\Rightarrow A'(x) = 3x^2 - 36x + 81 = 0$

         $\Rightarrow 3x^2 - 36x + 81 = 0$

         $\Rightarrow x^2 - 12x + 27 = 0$

         $\Rightarrow (x - 3)(x - 9) = 0$

         $\Rightarrow x = 3$ and $x = 9$ are critical numbers

      b. "Type b" ($f'(c)$ is undefined)

         Look for x-values that cause division by zero in $f'(x)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0) = (0)^3 - 18(0)^2 + 81(0) = 0$

      $A(3) = (3)^3 - 18(3)^2 + 81(3) = 108 \longleftarrow$ Abs Max Value

      $A(9) = (9)^3 - 18(9)^2 + 81(9) = 0$

5. Make sure that we've answered the original question.

   1. "Determine the value of x ..."

      $x = 3$