MTH 1126 Test #2 - 9am Class

Spring 2022

Pat Rossi Name ______________________

Instructions. Show CLEARLY how you arrive at your answers.

1. Use the "$f-g$" method to compute the area bounded by the graphs of $f(x)=x^2$ and $g(x)=3x^2-8$.

   First, graph the functions and find the points of intersection.

   $y=3x^2-8=x^2$

   i.e., $3x^2-8=x^2$

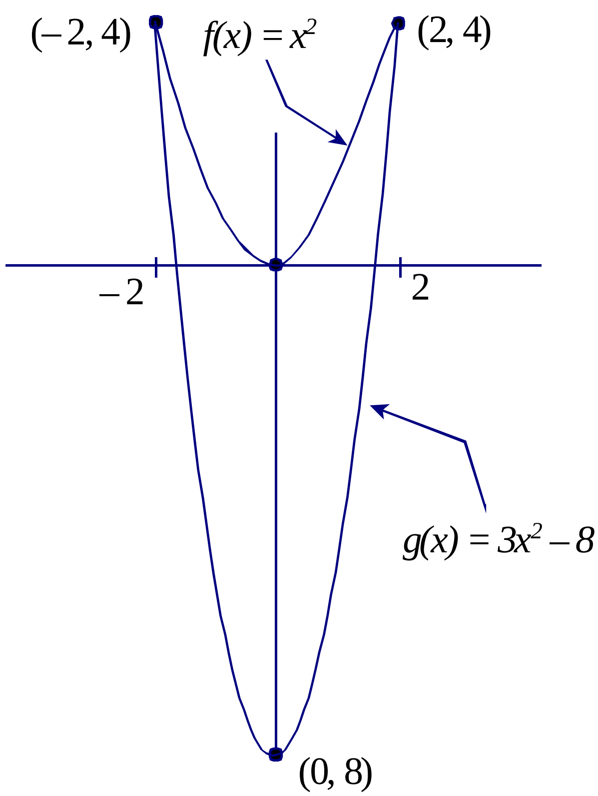
   $\Rightarrow 2x^2-8=0^2$

   $\Rightarrow x^2-4=0$

   $\Rightarrow (x+2)(x-2)=0$

   $\Rightarrow x=-2; \quad x=2$

   Points of intersection are $(-2,4)$ and $(2,4)$.



The bounded region spans the interval $[-2, 2]$ on the x-axis. Over this interval, $f(x) = x^2$ is greater than $g(x) = 3x^2 - 8$. Hence the area is given by:

$$A = \int_{-2}^{2} (x^2 - (3x^2 - 8))\, dx = \int_{-2}^{2} (8 - 2x^2)\, dx = \left[8x - \frac{2}{3}x^3\right]_{-2}^{2}$$

$$= \left(8(2) - \frac{2}{3}(2)^3\right) - \left(8(-2) - \frac{2}{3}(-2)^3\right) = \frac{64}{3}$$

i.e., bounded area $= \frac{64}{3}$

2. Find the area bounded by the graphs of $f(x) = x^2 - 4$ and $g(x) = -x^2 + 2x$. (Partition the appropriate interval, sketch the i$^{\text{th}}$ rectangle, build the Riemann Sum, derive the appropriate integral.)

Graph the functions and find the points of intersection.

To find the points of intersection, set the y-coordinates equal to one another and solve for x.

$y = x^2 - 4 = -x^2 + 2x$

$\Rightarrow 2x^2 - 2x - 4 = 0$

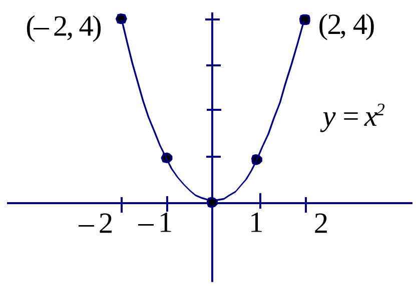
$\Rightarrow x^2 - x - 2 = 0$

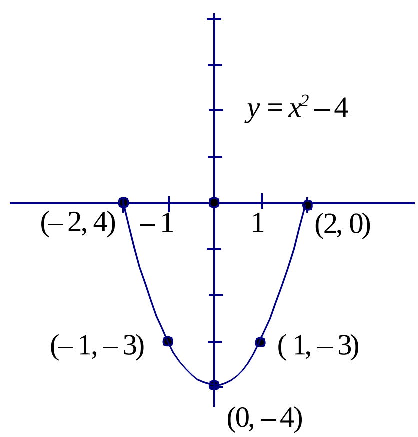
$\Rightarrow (x + 1)(x - 2) = 0.$

$\Rightarrow x = -1$; and $x = 2.$

Points of intersection: $(-1, -3)$ and $(2, 0)$.

To graph the functions, observe that the graph of $y = x^2 - 4$ can be obtained by taking the graph of $y = x^2$ and moving it down 4 units vertically.





To graph $y = -x^2 + 2x$, note that this is a quadratic (degree 2) equation. So its graph is a parabola. Since the coefficient of x^2 is negative, the graph "opens downward."

The x-intercepts of the graph are where $y = 0$. Therefore, to find the x-intercepts, we set $y = -x^2 + 2x = 0$.

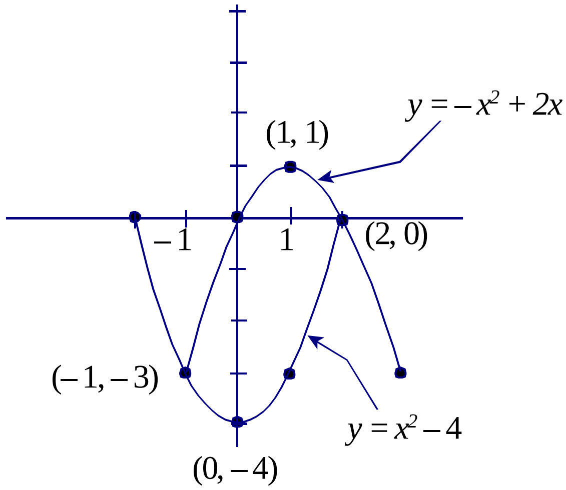
$\Rightarrow -x^2 + 2x = 0 \Rightarrow x^2 - 2x = 0 \Rightarrow x(x - 2) = 0 \Rightarrow x = 0,\ x = 2$ (x-intercepts).

Because the graph of a parabola is symmetric with respect to its vertex, the x-coordinate of the vertex is half way between the x-intercepts.

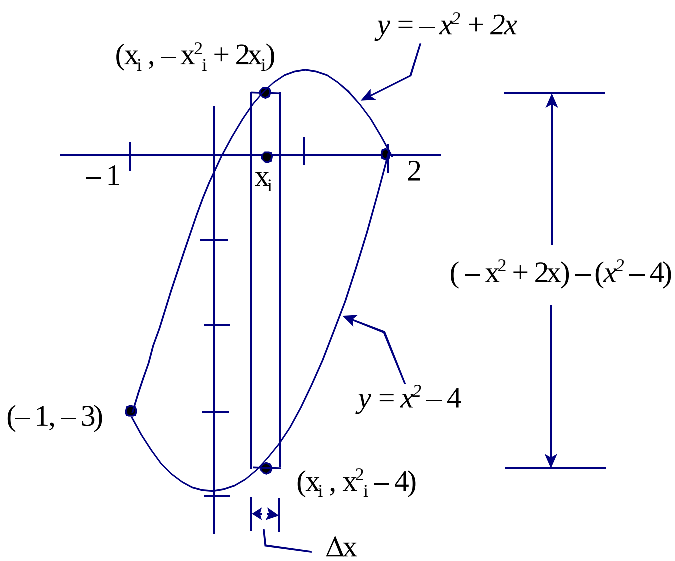
Therefore, the x-coordinate of the vertex is half way between $x = 0$ and $x = 2$.

i.e., the x-coordinate of the vertex is $x = 1$. (The vertex is $(1, 1)$.)

We have all that we need in order to graph the two functions.

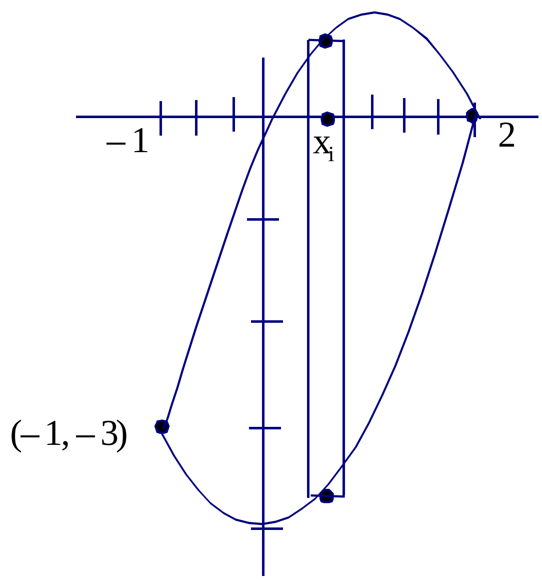


Having graphed the bounded region, we inscribe a thin rectangle of width $\triangle x$.



The area of the i^{th}. rectangle is $\underbrace{\left(\left(-x_i^2+2x_i\right)-\left(x_i^2-4\right)\right)}_{\text{height}}\cdot\underbrace{\Delta x}_{\text{width}}=\left(-2x_i^2+2x_i+4\right)\Delta x$

The rectangles span the interval $[-1,2]$ on the x-axis, so we will partition that interval into sub-intervals of length Δx.



The area of the bounded is approximately the sum of the areas of the rectangles.

$A \approx \sum_{i=1}^{n} \left(-2x_i^2 + 2x_i + 4\right) \Delta x$

Letting $\Delta x \to 0$, we get:

$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \left(-2x_i^2 + 2x_i + 4\right) \Delta x = \int_{x=-1}^{x=2} \left(-2x^2 + 2x + 4\right) dx = \left[-\frac{2}{3}x^3 + x^2 + 4x\right]_{-1}^{2}$

$= \left(-\frac{2}{3}\left(2\right)^3 + \left(2\right)^2 + 4\left(2\right)\right) - \left(-\frac{2}{3}\left(-1\right)^3 + \left(-1\right)^2 + 4\left(-1\right)\right) = 9$

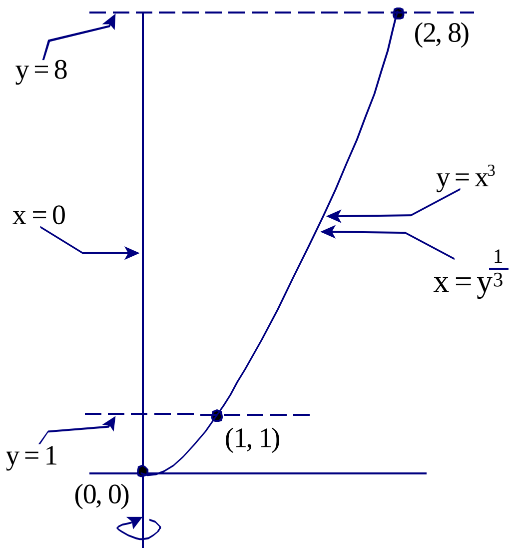
In retrospect, the limits of integration were the endpoints of the interval that we partitioned into sub-intervals.

Area $= 9$

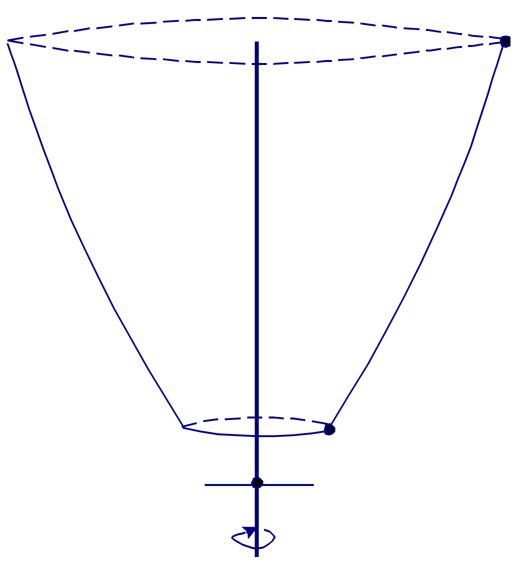
3. Use the "disc method" to compute the volume of the solid of revolution generated by revolving the region (in the first quadrant) bounded by the graphs of $x = y^{\frac{1}{3}}$, $y = 1$, $y = 8$, and the y-axis, about the y-axis. (For your information: the equation $y = x^3$ is the same as $x = y^{\frac{1}{3}}$.)

Use the "five step method" (partition the interval, sketch the i$^{\text{th}}$ rectangle, form the sum, take the limit)

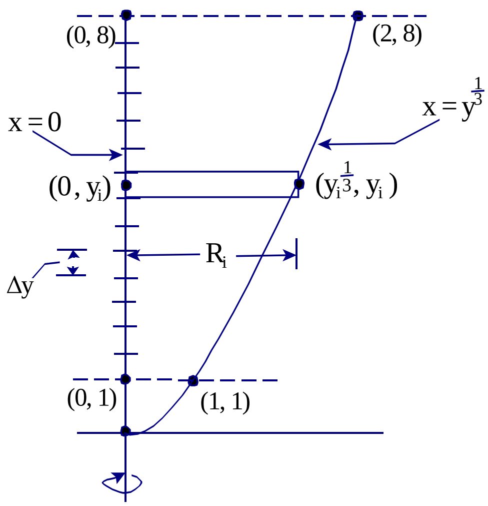
1) First, we'll graph the bounded region.



This is what the solid of revolution looks like:



2) Next, we sketch a rectangle of width Δx perpendicular (perpen-“disc”-ular) to the axis of revolution, and we partition the interval spanned by the rectangles.



3) Revolve the i[th] rectangle about the axis of revolution and compute the volume of the i[th] disc, Vol_i

$$Vol_i = \pi R_i^2 \Delta y = \pi \left(y_i^{\frac{1}{3}}\right)^2 \Delta y = \pi y_i^{\frac{2}{3}} \Delta y$$

4) Approximate the volume of the solid by adding up the volumes of the discs

$$Vol \approx \sum_{i=1}^{n} \pi y_i^{\frac{2}{3}} \Delta y$$

5) Let $\Delta y \to 0$

$$Vol = \lim_{\Delta y \to 0} \sum_{i=1}^{n} \pi y_i^{\frac{2}{3}} \Delta y = \int_{y=1}^{y=8} \pi y^{\frac{2}{3}} dy = \pi \int_{y=1}^{y=8} y^{\frac{2}{3}} dy$$

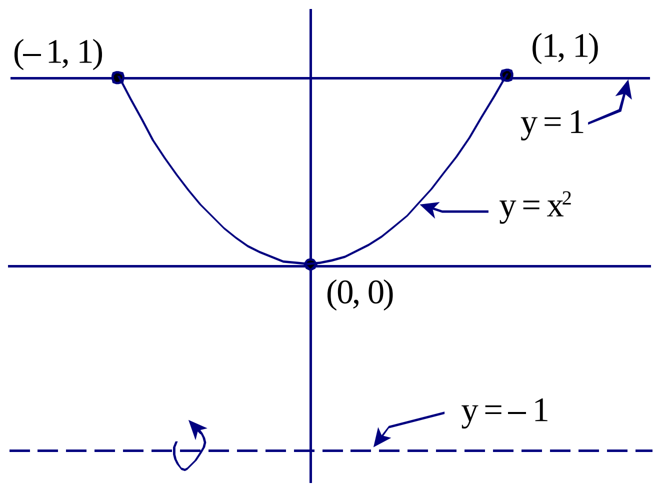
$$= \pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_{y=1}^{y=8} = \pi \left[\frac{3}{5}(8)^{\frac{5}{3}} - \frac{3}{5}(1)^{\frac{5}{3}}\right] = \pi \left[\frac{3}{5}(32) - \frac{3}{5}(1)\right] = \frac{93\pi}{5}$$

$$\boxed{Vol = \frac{93\pi}{5}}$$

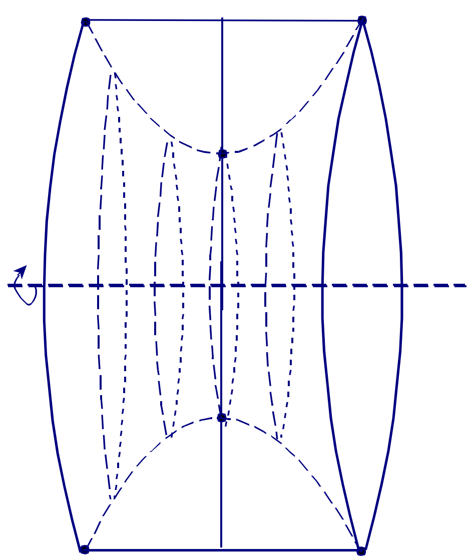
4. Use the “disc method” to compute the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = x^2$ and $y = 1$ about the line $y = -1$.

Use the “five step method” (partition the interval, sketch the i^{th} rectangle, form the sum, take the limit)

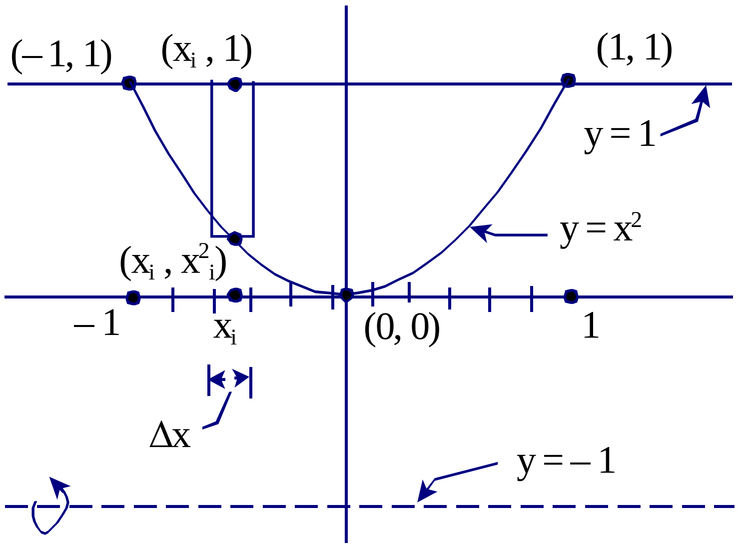
1) First, we’ll graph the bounded region.



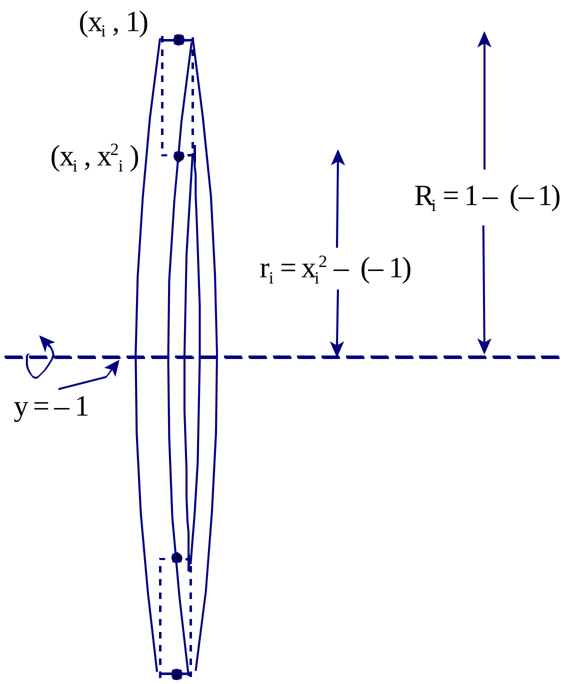
This is what the solid of revolution looks like:



2) Next, we sketch a rectangle of width Δx perpendicular (perpen-"disc"-ular) to the axis of revolution, and we partition the interval spanned by the rectangles.



3) Revolve the ith rectangle about the axis of revolution and compute the volume of the ith washer, Vol_i



$$Vol_i = Vol\, i^{th}\text{ Disc} - Vol\, i^{th}\text{ Hole} = \pi R_i^2\Delta x - \pi r_i^2\Delta x = \pi\left(1-(-1)\right)^2\Delta x - \pi\left(x_i^2-(-1)\right)^2\Delta x$$

$$= \pi\left(2\right)^2\Delta x - \pi\left(x_i^2+1\right)^2\Delta x = \pi 4\Delta x - \pi\left(x_i^4+2x_i^2+1\right)\Delta x = \pi\left(4-\left(x_i^4+2x_i^2+1\right)\right)\Delta x$$

$$= \pi\left(-x_i^4-2x_i^2+3\right)\Delta x$$

4) Approximate the volume of the solid by adding up the volumes of the washers

$Vol \approx \sum_{i=1}^{n} \pi\left(-x_i^4-2x_i^2+3\right)\Delta x$

5) Let $\Delta x \to 0$

$Vol = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi \left(-x_i^4 - 2x_i^2 + 3\right) \Delta x = \int_{x=-1}^{x=1} \pi \left(-x^4 - 2x^2 + 3\right) dx$

$$= \pi \left[-\frac{1}{5}x^5 - \frac{2}{3}x^3 + 3x\right]_{x=-1}^{x=1}$$

$$= \pi \left[-\frac{1}{5}(1)^5 - \frac{2}{3}(1)^3 + 3(1) - \left(-\frac{1}{5}(-1)^5 - \frac{2}{3}(-1)^3 + 3(-1)\right)\right]$$

$$= \pi \left[\frac{32}{15} - \left(-\frac{32}{15}\right)\right] = \frac{64\pi}{15}$$

Volume $= \frac{64\pi}{15}$