MTH 1126 – Test #1 – Solutions – 11am Class

SPRING 2022

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Show CLEARLY how you arrive at your answers

1. Compute: $\frac{d}{dx}\left[e^{\sqrt{3x}}\right] = \frac{d}{dx}\left[e^{(3x)^{\frac{1}{2}}}\right]$

$$\underbrace{\frac{d}{dx}\left[e^{(3x)^{\frac{1}{2}}}\right]}_{\frac{d}{dx}[e^u]} = \underbrace{e^{(3x)^{\frac{1}{2}}}}_{e^u} \cdot \underbrace{\frac{d}{dx}\left[(3x)^{\frac{1}{2}}\right]}_{\frac{du}{dx}} = e^{(3x)^{\frac{1}{2}}} \cdot \frac{1}{2}(3x)^{-\frac{1}{2}} \cdot 3 = \frac{3e^{(3x)^{\frac{1}{2}}}}{2(3x)^{\frac{1}{2}}}$$

$$\boxed{\begin{array}{l} \text{i.e., } \frac{d}{dx}\left[e^{(3x)^{\frac{1}{2}}}\right] = \frac{3e^{(3x)^{\frac{1}{2}}}}{2(3x)^{\frac{1}{2}}} \\ \\ \text{Or: } \frac{d}{dx}\left[e^{(3x)^{\frac{1}{2}}}\right] = \frac{3e^{\sqrt{3x}}}{2\sqrt{3x}} \end{array}}$$

2. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{e^x+x^2}{x^3}}\right)\right] =$

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{e^x+x^2}{x^3}}\right)\right] = \frac{d}{dx}\left[\ln\left(\left(\frac{e^x+x^2}{x^3}\right)^{\frac{1}{2}}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{e^x+x^2}{x^3}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(e^x+x^2\right) - \ln\left(x^3\right)\right)\right]$$

$$= \frac{1}{2}\left(\underbrace{\frac{1}{e^x+x^2}}_{\frac{i}{u}} \cdot \underbrace{e^x+2x}_{\frac{du}{dx}} - \underbrace{\frac{1}{x^3}}_{\frac{i}{u}} \cdot \underbrace{3x^2}_{\frac{du}{dx}}\right) = \frac{1}{2}\left(\frac{e^x+2x}{e^x+x^2} - \frac{3x^2}{x^3}\right) = \frac{1}{2}\left(\frac{e^x+2x}{e^x+x^2} - \frac{3}{x}\right) = \frac{e^x+2x}{2e^x+2x^2} - \frac{3}{2x}$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{e^x+x^2}{x^3}}\right)\right] = \frac{1}{2}\left(\frac{e^x+2x}{e^x+x^2} - \frac{3}{x}\right)$

Or: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{e^x+x^2}{x^3}}\right)\right] = \frac{e^x+2x}{2e^x+2x^2} - \frac{3}{2x}$

(Alternative Solution Appears on the Following Page)

Alternative Solution:

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{e^x+x^2}{x^3}}\right)\right] = \frac{d}{dx}\left[\underbrace{\ln\left[\left(\frac{e^x+x^2}{x^3}\right)^{\frac{1}{2}}\right]}_{\ln(u)}\right] = \underbrace{\frac{1}{\left(\frac{e^x+x^2}{x^3}\right)^{\frac{1}{2}}}}_{\frac{1}{u}} \cdot \underbrace{\left(\frac{d}{dx}\left[\left(\frac{e^x+x^2}{x^3}\right)^{\frac{1}{2}}\right]\right)}_{\frac{du}{dx}}$$

$$= \left(\frac{e^x+x^2}{x^3}\right)^{-\frac{1}{2}} \cdot \frac{1}{2}\left(\frac{e^x+x^2}{x^3}\right)^{-\frac{1}{2}} \underbrace{\frac{(e^x+2x)\left(x^3\right) - \left(3x^2\right)\left(e^x+x^2\right)}{\left(x^3\right)^2}}_{\text{Quotient Rule}}$$

$$= \left(\frac{x^3}{e^x+x^2}\right)^{\frac{1}{2}} \cdot \frac{1}{2}\left(\frac{x^3}{e^x+x^2}\right)^{\frac{1}{2}} \frac{(e^x+2x)\left(x^3\right) - \left(3x^2\right)\left(e^x+x^2\right)}{x^6}$$

$$= \frac{1}{2}\left(\frac{x^3}{e^x+x^2}\right) \frac{(e^x+2x)\left(x^3\right) - \left(3x^2\right)\left(e^x+x^2\right)}{x^6} = \frac{1}{2}\left(\frac{1}{e^x+x^2}\right) \frac{(e^x+2x)\left(x^3\right) - \left(3x^2\right)\left(e^x+x^2\right)}{x^3} = \frac{1}{2}\left(\frac{1}{e^x+x^2}\right) \frac{(e^x+2x)x - 3\left(e^x+x^2\right)}{x}$$
$$\frac{(x-3)e^x - x^2}{2x(e^x+x^2)}$$

$$\boxed{\begin{gathered}\text{i.e., } \frac{d}{dx}\left[\ln\left(\sqrt{\frac{e^x+x^2}{x^3}}\right)\right] = \frac{1}{2}\left(\frac{x^3}{e^x+x^2}\right) \frac{(e^x+2x)\left(x^3\right) - \left(3x^2\right)\left(e^x+x^2\right)}{x^6} = \frac{(x-3)e^x - x^2}{2x(e^x+x^2)} \\[2em] \text{Or: } \frac{d}{dx}\left[\ln\left(\sqrt{\frac{e^x+x^2}{x^3}}\right)\right] = \frac{(x-3)e^x - x^2}{2x(e^x+x^2)}\end{gathered}}$$

3. Compute: $\int e^{\left(8x^3+6x^2\right)}\left(8x^2+4x\right)\,dx =$

(a) 1. Is u-sub appropriate?

a. Is there a composite function?

Yes. $e^{\left(8x^3+6x^2\right)}$

Let $u = 8x^3 + 6x^2$

b. Is there an "approximate function/derivative pair"?

Yes. $\underbrace{\left(8x^3+6x^2\right)}_{\text{function}} \rightarrow \underbrace{\left(8x^2+4x\right)}_{\text{deriv}}$

Let $u = (8x^3 + 6x^2)$

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & 8x^3+6x^2 \\ \Rightarrow & \frac{du}{dx} & = & 24x^2+12x \\ \Rightarrow & du & = & (24x^2+12x)\,dx \\ \Rightarrow & \frac{1}{3}du & = & (8x^2+4x)\,dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{e^{\left(8x^3+6x^2\right)}}_{e^u}\underbrace{\left(8x^2+4x\right)\,dx}_{\frac{1}{3}du} = \int e^u \tfrac{1}{3}du = \tfrac{1}{3}\int e^u\,du$$

4. Integrate in terms of u

$\frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C$

5. Re-write in terms of x

$$\int e^{\left(8x^3+6x^2\right)}\left(8x^2+4x\right)\,dx = \underbrace{\frac{1}{3}e^{\left(8x^2+4x\right)} + C}_{\frac{1}{3}e^u+C}$$

$$\boxed{\text{i.e., } \int e^{\left(8x^3+6x^2\right)}\left(8x^2+4x\right)\,dx = \tfrac{1}{3}e^{\left(8x^2+4x\right)} + C}$$

4. Compute: $\int \frac{3x^2-1}{(4x^3-4x+5)^5}\,dx = \int \frac{1}{(4x^3-4x+5)^5}\left(3x^2-1\right)\,dx = \int \left(4x^3-4x+5\right)^{-5}\left(3x^2-1\right)\,dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

      Yes. $\left(4x^3-4x+5\right)^{-5}$

      Let $u = (4x^3-4x+5)$ i.e., "Let $u =$ the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

      Yes. $\underbrace{\left(4x^3-4x+5\right)}_{\text{function}} \rightarrow \underbrace{3x^2-1}_{\text{deriv}}$

      Let $u = (4x^3-4x+5)$ i.e., "Let $u =$ 'the function'"

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & 4x^3-4x+5 \\ \Rightarrow & \frac{du}{dx} & = & 12x^2-4 \\ \Rightarrow & du & = & (12x^2-4)\,dx \\ \Rightarrow & \frac{1}{4}du & = & (3x^2-1)\,dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\int \left(4x^3-4x+5\right)^{-5}}_{u^{-5}} \underbrace{\left(3x^2-1\right)\,dx}_{\frac{1}{4}du} = \int u^{-5}\tfrac{1}{4}du = \tfrac{1}{4}\int u^{-5}\,du$$

4. Integrate in terms of u

$\frac{1}{4}\int u^{-5}\,du = \frac{1}{4}\frac{u^{-4}}{-4} + C = -\frac{1}{16}u^{-4} + C$

5. Re-write in terms of x

$$\int \frac{3x^2-1}{(4x^3-4x+5)^5}\,dx = \underbrace{-\frac{1}{16}\left(4x^3-4x+5\right)^{-4} + C}_{-\frac{1}{16}u^{-4}+C}$$

$$\boxed{\text{i.e., } \int \frac{3x^2-1}{(4x^3-4x+5)^5}\,dx = -\frac{1}{16}\left(4x^3-4x+5\right)^{-4} + C}$$

5. Compute: $\int \frac{2x^2+x+1}{(4x^3+3x^2+6x)}\,dx = \int \frac{1}{(4x^3+3x^2+6x)}\left(2x^2+x+1\right)dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes. $\frac{1}{(4x^3+3x^2+6x)} = \left(4x^3+3x^2+6x\right)^{-1}$

   Let $u = (4x^3+3x^2+6x)$ i.e., "Let $u =$ the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

   Yes. $\underbrace{\left(4x^3+3x^2+6x\right)}_{\text{function}} \to \underbrace{\left(2x^2+x+1\right)}_{\text{deriv}}$

   Let $u = (4x^3+3x^2+6x)$ i.e., "Let $u =$ 'the function'"

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & 4x^3+3x^2+6x \\ \Rightarrow & \frac{du}{dx} & = & 12x^2+6x+6 \\ \Rightarrow & du & = & (12x^2+6x+6)\,dx \\ \Rightarrow & \frac{1}{6}du & = & (2x^2+x+1)\,dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\int \frac{1}{(4x^3+3x^2+6x)}}_{\frac{1}{u}} \underbrace{\left(2x^2+x+1\right)dx}_{\frac{1}{6}du} = \int \frac{1}{u}\frac{1}{6}du = \frac{1}{6}\int \frac{1}{u}\,du$$

4. Integrate in terms of u

$\frac{1}{6}\int \frac{1}{u}\,du = \frac{1}{6}\ln|u| + C$

5. Re-write in terms of x

$$\int \frac{2x^2+x+1}{(4x^3+3x^2+6x)}\,dx = \underbrace{\frac{1}{6}\ln\left|4x^3+3x^2+6x\right| + C}_{\frac{1}{6}\ln|u|+C}$$

$$\boxed{\text{i.e., } \int \frac{2x^2+x+1}{(4x^3+3x^2+6x)}\,dx = \frac{1}{6}\ln|4x^3+3x^2+6x| + C}$$

6. Compute: $\frac{d}{dx}\left[\operatorname{arcsec}\left(e^{x^2}\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\operatorname{arcsec}\left(e^{x^2}\right)\right]}_{\frac{d}{dx}[\operatorname{arcsec}(u)]} = \underbrace{\frac{1}{|e^{x^2}|\sqrt{\left(e^{x^2}\right)^2 - 1}}}_{\frac{1}{|u|\sqrt{u^2-1}}} \cdot \underbrace{e^{x^2} \cdot 2x}_{\frac{du}{dx}} = \frac{e^{x^2}2x}{e^{x^2}\sqrt{e^{2x^2}-1}} = \frac{2x}{\sqrt{e^{2x^2}-1}}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[\operatorname{arcsec}\left(e^{x^2}\right)\right] = \frac{2x}{\sqrt{e^{2x^2}-1}}}$$

7. Compute: $\int \frac{1}{x^2\sqrt{9x^4-4}} x\,dx$

This appears to fit the form: $\int \frac{1}{u\sqrt{u^2-a^2}} du$

If our conjecture is correct, then $\sqrt{u^2-a^2} = \sqrt{9x^4-4}$

$$\sqrt{\underbrace{u^2}-\underbrace{a^2}} = \sqrt{\underbrace{9x^4}-\underbrace{4}}$$

$$\boxed{\begin{aligned} \Rightarrow \quad & a^2 = 4 \\ & a = 2 \\ \Rightarrow \quad & u^2 = 9x^4 \\ & u = 3x^2 \\ & \tfrac{1}{3}u = x^2 \\ \Rightarrow \quad & \tfrac{du}{dx} = 6x \\ & du = 6x dx \\ & \tfrac{1}{6}du = x dx \end{aligned}}$$

$$\int \frac{1}{\underbrace{x^2}\underbrace{\sqrt{9x^4-4}}}\underbrace{xdx} = \int \frac{1}{\underbrace{\left(\frac{1}{3}u\right)}\underbrace{\sqrt{u^2-a^2}}}\underbrace{\left(\frac{1}{6}du\right)}$$

3. Analyze in terms of u and du.

$$\int \frac{1}{x^2\sqrt{9x^4-4}} x\,dx = \int \frac{1}{x^2\sqrt{(3x^2)^2-2^2}} xdx = \int \frac{1}{\left(\frac{1}{3}u\right)\sqrt{u^2-a^2}} \left(\frac{1}{6}du\right) = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-a^2}} du$$

4. Integrate

$$\frac{1}{2}\int \frac{1}{u\sqrt{u^2-a^2}} du = \frac{1}{2}\left(\frac{1}{a}\text{arcsec}\left(\frac{|u|}{a}\right)\right) + C = \frac{1}{2a}\text{arcsec}\left(\frac{|u|}{a}\right) + C$$

5. Re-express in terms of x

$$\int \frac{1}{x^2\sqrt{9x^4-4}} x\,dx = \underbrace{\frac{1}{4}\text{arcsec}\left(\frac{|3x^2|}{2}\right) + C}_{\frac{1}{2a}\text{arcsec}\left(\frac{|u|}{a}\right)+C}$$

$$\boxed{\int \frac{1}{x^2\sqrt{9x^4-4}} x\,dx = \frac{1}{4}\text{arcsec}\left(\frac{|3x^2|}{2}\right) + C}$$

8. Compute: $\frac{d}{dx}\left[\sin^{-1}\left(\cos\left(x\right)\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\sin^{-1}\left(\cos\left(x\right)\right)\right]}_{\frac{d}{dx}[\tan^{-1}(u)]} = \underbrace{\frac{1}{\sqrt{1-\left(\cos\left(x\right)\right)^2}}}_{\frac{1}{\sqrt{1-u^2}}} \cdot \underbrace{\left(-\sin\left(x\right)\right)}_{\frac{du}{dx}} = -\frac{\sin(x)}{\sqrt{1-\cos^2(x)}}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[\sin^{-1}\left(\cos\left(x\right)\right)\right] = -\frac{\sin(x)}{\sqrt{1-\cos^2(x)}}}$$

9. Compute: $\int \frac{x}{9+16x^4}\,dx = \int \frac{1}{9+16x^4} x\,dx$

1. a. Is there a composite function?

   Yes. $\frac{1}{9+16x^4} = \underbrace{(9+16x^4)}_{\text{inner}}{}^{-1}$

   Let $u = 9 + 16x^4$

   Is there an "approximate function/derivative pair"?

   There does not appear to be an "approximate function/derivative pair."

   Proceeding solely on the strength of Part a, we continue, aware of the possibility that u-substitution might not work.

2. Compute du

   $$\boxed{\begin{array}{lrcl} & u & = & 9+16x^4 \\ \Rightarrow & \frac{du}{dx} & = & 64x^3 \\ \Rightarrow & du & = & 64x^3 dx \\ \Rightarrow & \frac{1}{64x^2} du & = & x\,dx \end{array}}$$

3. Analyze in terms of u and du.

   $$\int \underbrace{\frac{1}{9+16x^4}}_{\frac{1}{u}} \cdot \underbrace{x\,dx}_{\frac{1}{64x^2}du} =$$

   Since we cannot analyze the integral solely in terms of u and du, u-substitution alone will not work in this case.

   We must try to get our integral to fit a different form. (See Next Page)

Exercise 9 Continued . . .

$\int \frac{1}{9+16x^4} x\, dx$ compare to: $\int \frac{1}{a^2+u^2} du$

$$a^2 + u^2 = 9 + 16x^4$$

If this comparison is correct, then:

	$a^2 = 9$
$\Rightarrow$	$a = 3$
	$u^2 = 16x^4$
$\Rightarrow$	$u = 4x^2$
$\Rightarrow$	$\frac{du}{dx} = 8x$
$\Rightarrow$	$du = 8xdx$
$\Rightarrow$	$\frac{1}{8}du = xdx$

$$\int \frac{1}{9+16x^4} xdx = \int \frac{1}{a^2+u^2}\left(\frac{1}{8}du\right)$$

Now analyze the integral in terms of u and du.

$\int \frac{1}{9+16x^4} x\, dx = \int \frac{1}{3^2+(4x^2)^2} x\, dx = \int \frac{1}{a^2+u^2} \frac{1}{8} du = \frac{1}{8} \int \frac{1}{a^2+u^2} du$

3. Integrate:

$\frac{1}{8} \int \frac{1}{a^2+u^2} du = \frac{1}{8} \cdot \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C = \frac{1}{8} \cdot \frac{1}{3} \arcsin\left(\frac{4x^2}{3}\right) + C = \frac{1}{24} \arcsin\left(\frac{4x^2}{3}\right) + C$

$$\boxed{\text{i.e., } \int \frac{1}{9+16x^4} x\, dx = \frac{1}{24} \arcsin\left(\frac{4x^2}{3}\right) + C}$$

10. $z = \cos(\operatorname{arcsec}(2x))$ Re-write this equation as an equivalent algebraic equation.

Let $w = \operatorname{arcsec}(2x)$

Then "w is the angle whose secant is $2x$."

i.e., $\sec(w) = 2x$

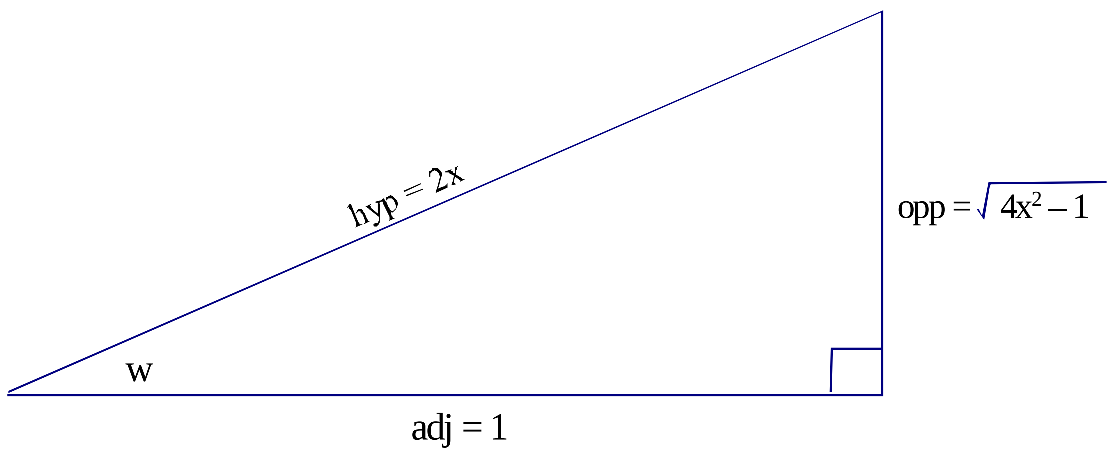
Draw a right triangle that depicts this relationship.

i.e., $\sec(w) = \frac{2x}{1} = \frac{\text{hyp}}{\text{adj}}$

$opp^2 = hyp^2 - adj^2 = (2x)^2 - 1^2 = 4x^2 - 1$

i.e., $opp^2 = 4x^2 - 1$

$\Rightarrow opp = \sqrt{4x^2 - 1}$



We want $z = \cos(\operatorname{arcsec}(2x))$

But since $w = \operatorname{arcsec}(2x)$,

$\Rightarrow z = \cos(w)$

$\Rightarrow z = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2x}$

$\boxed{\text{i.e., } z = \frac{1}{2x}}$

Extra: Wow! 10 points (All or nothing)

Compute: $\int \frac{1}{e^{-x}+e^{x}} dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

      Yes. $\frac{1}{e^{-x}+e^{x}} = \left(e^{-x}+e^{x}\right)^{-1}$

      Let $u = e^{-x}+e^{x}$ i.e., "Let $u =$ the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

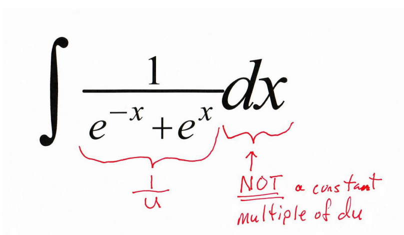
      ???. I sure don't see one! I don't see the derivative of $e^{-x}+e^{x}$ anywhere. Not only that, I don't see the derivative of e^{x} anywhere else in the integrand, nor do I see the derivative of e^{-x} anywhere else in the integrand.

      On the strength of criterion a. alone, if we let $u = e^{-x}+e^{x}$, then we have:

      $\frac{du}{dx} = \frac{d}{dx}\left[e^{-x}+e^{x}\right] = -e^{-x}+e^{x}$

      $du = \left(-e^{-x}+e^{x}\right) dx$

      This yields:



      We have $\frac{1}{u}$, but we don't have a constant multiple of du anywhere!

      As some of my older relatives would have said: "This ain' goan work!"

      Somehow, through algebraic trickery, we have to create du.

      We have e^{x} in the integrand. Somewhere else in the integrand, we should find its derivative, e^{x}. But there isn't one! Just "on a hunch," maybe we could put a factor of e^{x} somewhere else in the integrand and compensate by dividing by e^{x}.

      $\int \frac{1}{e^{x}} \cdot \frac{1}{e^{-x}+e^{x}} \cdot e^{x} dx$

      This yields:

      $\int \left(\frac{1}{e^{x}} \frac{1}{e^{-x}+e^{x}}\right) e^{x} dx = \int \left(\frac{1}{e^{x}\left(e^{-x}+e^{x}\right)}\right) e^{x} dx = \int \left(\frac{1}{e^{x} \cdot e^{-x}+e^{x} \cdot e^{x}}\right) e^{x} dx = \int \frac{1}{1+\left(e^{x}\right)^{2}} e^{x} dx$

      OMG! We're onto something here!

      $\int \frac{1}{e^{-x}+e^{x}} dx = \int \frac{1}{1+\left(e^{x}\right)^{2}} e^{x} dx$

This fits the form: $\int \frac{1}{a^2+u^2} du$

	$a^2 = 1$
$\Rightarrow$	$a = 1$
	$u^2 = (e^x)^2$
$\Rightarrow$	$u = e^x$
$\Rightarrow$	$\frac{du}{dx} = e^x$
$\Rightarrow$	$du = e^x dx$

$$\int \frac{1}{1+\left(e^x\right)^2} e^x dx = \int \frac{1}{a^2+u^2} du$$

$$\int \frac{1}{e^{-x}+e^x} dx = \int \frac{1}{1+(e^x)^2} e^x dx = \int \frac{1}{a^2+u^2} du = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C = \frac{1}{1} \arctan\left(\frac{e^x}{1}\right) + C$$

$$= \arctan\left(e^x\right) + C$$

$$\boxed{\text{i.e., } \int \frac{1}{e^{-x}+e^x} dx = \arctan\left(e^x\right) + C}$$