MTH 3311 Test #3 - Solutions

Spring 2018

Pat Rossi Name ______________________________

Show CLEARLY how you arrive at your answers

1. Find the general solution of the equation: $y'' - 4y' + 4y = 12x^2 - 34$

   First, find the solution to the complementary equation $y'' - 4y' + 4y = 0$

   The auxiliary equation is $m^2 - 4m + 4 = 0$

   $\Rightarrow m^2 - 4m + 4 = 0 \Rightarrow (m-2)(m-2) = 0 \Rightarrow m_1 = 2$ and $m_2 = 2$ (Double Root)

   In this case, $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{2x} + c_2 e^{2x}$ won't work, because the two terms are essentially the same term.

   To remedy the situation, we multiply the second term by x.

   This yields: $y_c = c_1 e^{2x} + c_2 x e^{2x}$

   For the particular solution, we imagine that $y_p = Ax^2 + Bx + C$

   $y_p' = 2Ax + B$

   $y_p'' = 2A$

   To find A, B and C, we plug these into the original equation, $y'' - 4y' + 4y = 12x^2 - 34$.

   This yields:

   $$\underbrace{2A}_{y''} - \underbrace{4(2Ax + B)}_{4y'} + \underbrace{4\left(Ax^2 + Bx + C\right)}_{4y} = 12x^2 - 34$$

   $\Rightarrow 4Ax^2 - (8A - 4B)x + (2A - 4B + 4C) = 12x^2 - 34$

   Comparing the coefficients of the different powers of x, we get:

   $$\begin{array}{lllllllll} 4A & & & & & = & 12 & & \text{(Eq. 1)} \\ -8A & + & 4B & & & = & 0 & & \text{(Eq. 2)} \\ 2A & - & 4B & + & 4C & = & -34 & & \text{(Eq. 3)} \end{array}$$

   From Eq. 1, we get: $A = 3$

   Plugging $A = 3$ into Eq. 2, we get: $-8(3) + 4B = 0 \Rightarrow 4B = 24 \Rightarrow B = 6$

   Plugging $A = 3$ and $B = 6$ into Eq. 3, we get: $2(3) - 4(6) + 4C = -34 \Rightarrow 4C = -16 \Rightarrow C = -4$

   Hence, $y_p = 3x^2 + 6x - 4$

   The solution to the original equation is: $y = y_p + y_c$

   $$\boxed{\Rightarrow y = 3x^2 + 6x - 4 + c_1 e^{2x} + c_2 x e^{2x}}$$

2. Find the general solution of the equation: $x^2y'' + 2xy' - 6y = 24x^3$

First, find the solution to the complementary equation $x^2y'' + 2xy' - 6y = 0$

Our strategy is to seek solutions of the form:

$y = x^\lambda$

$\Rightarrow y' = \lambda x^{\lambda-1}$

$\Rightarrow y'' = \lambda(\lambda-1)x^{\lambda-2}$

Plugging these into the complementary equation $x^2y'' + 2xy' - 6y = 0$, we have:

$x^2\lambda(\lambda-1)x^{\lambda-2} + 2x\lambda x^{\lambda-1} - 6x^\lambda = 0$

$\Rightarrow \lambda(\lambda-1)x^\lambda + 2\lambda x^\lambda - 6x^\lambda = 0$

$\Rightarrow \lambda(\lambda-1) + 2\lambda - 6 = 0$

$\Rightarrow \lambda^2 + \lambda - 6 = 0$

$\Rightarrow (\lambda+3)(\lambda-2) = 0$

$\Rightarrow \lambda_1 = -3;\ \lambda_2 = 2$

Our complementary solution is:

$y_c = c_1x^{\lambda_1} + c_2x^{\lambda_2} = c_1x^{-3} + c_2x^2$

Next, we find our particular solution

Since the right hand side of the equation is the polynomial $24x^3$, we guess that the particular solution is a polynomial having only terms of the same degree that appear on the right hand side of the original equation. (Note: In this respect, the Method of Undetermined Coefficients for Euler Equations differs from the Method of Undetermined Coefficients for Equations with Constant Coefficients.)

Thus, we guess that:

$y = Ax^3$

$\Rightarrow y' = 3Ax^2$

$\Rightarrow y'' = 6Ax$

To find A, B and C, we plug these into the original equation, $x^2y'' + 2xy' - 6y = 24x^3$.

This yields:

$6Ax^3 + 6Ax^3 - 6Ax^3 = 24x^3$

$\Rightarrow 6Ax^3 = 24x^3$

$\Rightarrow A = 4$

Hence, $y_p = 4x^3$

The solution to the original equation is: $y = y_p + y_c$

$$\boxed{\Rightarrow y = 4x^3 + c_1x^{-3} + c_2x^2}$$

3. Find the general solution of the equation: $y'' + y = \csc(x)$

First, find the solution to the complementary equation $y'' + y = 0$

The auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i \Rightarrow m_1 = i$ and $m_2 = -i$

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{ix} + c_2 e^{-ix} = A\cos(x) + B\sin(x)$

To remedy the situation, we multiply the second term by x.

i.e., $y_c = A\cos(x) + B\sin(x)$

Next, we find the particular solution.

Since the right hand side of the original equation is not a linear combination of sines, cosines, exponentials, and polynomials, the method of Undetermined Coefficients won't work.

Therefore we must use Variation of Parameters.

We convert the complementary solution into the general solution: $y = A(x)\cos(x) + B(x)\sin(x)$

We have two restrictions that we can impose on this pair of functions.

The first restriction that we impose is that the pair $A(x), B(x)$ actually does make the equation

$y = A(x)\cos(x) + B(x)\sin(x)$ the general solution.

$y = A(x)\cos(x) + B(x)\sin(x)$

$\Rightarrow y' = A(x)(-\sin(x)) + A'(x)\cos(x) + B(x)\cos(x) + B'(x)\sin(x)$

To simplify this expression, we impose our **second restriction:**

$A'(x)\cos(x) + B'(x)\sin(x) = 0 \qquad \text{(Eq.1)}$

$\Rightarrow y' = -A(x)\sin(x) + B(x)\cos(x)$

$\Rightarrow y'' = -A(x)\cos(x) - A'(x)\sin(x) + B(x)(-\sin(x)) + B'(x)\cos(x)$

i.e., $y'' = -A(x)\cos(x) - A'(x)\sin(x) - B(x)\sin(x) + B'(x)\cos(x)$

We plug these into the original equation: $y'' + y = \csc(x)$

$\Rightarrow (-A(x)\cos(x) - A'(x)\sin(x) - B(x)\sin(x) + B'(x)\cos(x)) + (A(x)\cos(x) + B(x)\sin(x)) = \csc(x)$

$\Rightarrow -A'(x)\sin(x) + B'(x)\cos(x) = \csc(x)$

To eliminate one of these terms, we will use this equation in combination with Eq.1

$$\begin{array}{rllcll} & -A'(x)\sin(x) & + & B'(x)\cos(x) & = & \csc(x) \\ +\tan(x) & [A'(x)\cos(x) & + & B'(x)\sin(x)] & = & \tan(x)[0] \\ \hline & & & B'(x)\left(\frac{\sin^2(x)}{\cos(x)} + \cos(x)\right) & = & \csc(x) \end{array} \qquad \text{(We multiplied Eq. 1 by } \tan(x)\text{)}$$

i.e., $B'(x)\left(\frac{\sin^2(x)}{\cos(x)} + \cos(x)\right) = \csc(x)$

$\Rightarrow B'(x)\left(\sin^2(x)+\cos^2(x)\right)=\cot(x)$ (We multiplied both sides by $\cos(x)$)

$\Rightarrow B'(x)=\cot(x)$

$\Rightarrow B(x)=\int\cot(x)\,dx=\ln|\sin(x)|+c_1$

i.e., $B(x)=\ln|\sin(x)|+c_1$

To find $A(x)$, we substitute $B'(x)=\cot(x)$ into Eq. 1 (Our second restriction)

$\Rightarrow A'(x)\cos(x)+\underbrace{\cot(x)}_{B'(x)}\sin(x)=0$

$\Rightarrow A'(x)\cos(x)+\cos(x)=0$

$\Rightarrow A'(x)+1=0$

$\Rightarrow A'(x)=-1$

$\Rightarrow A(x)=\int(-1)\,dx=-x+c_2$

i.e., $A(x)=-x+c_2$

Our general solution is $y=A(x)\cos(x)+B(x)\sin(x)=(-x+c_2)\cos(x)+(\ln|\sin(x)|+c_1)\sin(x)$

$$\boxed{y=(-x+c_2)\cos(x)+(\ln|\sin(x)|+c_1)\sin(x)}$$