MTH 1125 Test #1 - Solutions

SUMMER 2012

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\lim_{x\to 2} \frac{x^2+2x+4}{x^3-2x+4} =$

   (a) 1. Try Plugging in:

   $\lim_{x\to 2} \frac{x^2+2x+4}{x^3-2x+4} = \frac{(2)^2+2(2)+4}{(2)^3-2(2)+4} = \frac{12}{8} = \frac{3}{2}$

   $\boxed{\text{i.e., } \lim_{x\to 2} \frac{x^2+2x+4}{x^3-2x+4} = \frac{3}{2}}$

2. Compute: $\lim_{x\to 3} \frac{x^2-2x-3}{x^2-x-6} =$

   (a) 1. Try Plugging in:

   $\lim_{x\to 3} \frac{x^2-2x-3}{x^2-x-6} = \frac{(3)^2-2(3)-3}{(3)^2-(3)-6} = \frac{0}{0}$ No Good - Zero Divide!

   2. Try Factoring and Cancelling:

   $\lim_{x\to 3} \frac{x^2-2x-3}{x^2-x-6} = \lim_{x\to 3} \frac{(x+1)(x-3)}{(x+2)(x-3)} = \lim_{x\to 3} \frac{(x+1)}{(x+2)} = \frac{(3)+1}{(3)+2} = \frac{4}{5}$

   $\boxed{\text{i.e., } \lim_{x\to 3} \frac{x^2-2x-3}{x^2-x-6} = \frac{4}{5}}$

3. Compute: $\lim_{x\to -\infty} \frac{5x^3+8x^2+5}{5x^4-4x^3+5x^2+x} =$

   $\lim_{x\to -\infty} \frac{5x^3+8x^2+5}{5x^4-4x^3+5x^2+x} = \lim_{x\to -\infty} \frac{5x^3}{5x^4} = \lim_{x\to -\infty} \frac{1}{x} = 0$

   $\boxed{\text{i.e., } \lim_{x\to -\infty} \frac{5x^3+8x^2+5}{5x^4-4x^3+5x^2+x} = 0}$

4. Compute: $\lim_{x\to -3} \frac{x+5}{x^2+x-6} =$

(a) 1. Try Plugging in:

$\lim_{x\to -3} \frac{x+5}{x^2+x-6} = \frac{(-3)+5}{(-3)^2+(-3)-6} = \frac{2}{0}$ No Good - Zero Divide!

2. Try Factoring and Cancelling:

No Good!. "Factoring and Cancelling" only works when Step #1 yields $\frac{0}{0}$.

3. Analyze the one-sided limits:

$\lim_{x\to -3^-} \frac{x+5}{x^2+x-6} = \lim_{x\to -3^-} \frac{x+5}{(x+3)(x-2)} = \frac{2}{(-\varepsilon)(-5)} = \frac{\left(-\frac{2}{5}\right)}{(-\varepsilon)} = +\infty$

$$\boxed{\begin{array}{ll} & x \to -3^- \\ \Rightarrow & x < -3 \\ \Rightarrow & x+3 < 0 \end{array}}$$

$\lim_{x\to -3^+} \frac{x+5}{x^2-x-6} = \lim_{x\to -3^+} \frac{x+5}{(x+3)(x-2)} = \frac{2}{(+\varepsilon)(-5)} = \frac{\left(-\frac{2}{5}\right)}{(+\varepsilon)} = -\infty$

$$\boxed{\begin{array}{ll} & x \to -3^+ \\ \Rightarrow & x > -3 \\ \Rightarrow & x+3 > 0 \end{array}}$$

Since the one-sided limits are not equal, $\lim_{x\to -3} \frac{x+5}{x^2+x-6}$ **Does Not Exist!**

5. $f(x) = 10x^4 + 5x^3 + 20x^2 + 15x + 40\sqrt{x} + 5$; Compute: $f'(x)$.

Rewrite: $f(x) = 10x^4 + 5x^3 + 20x^2 + 15x + 40x^{\frac{1}{2}} + 5$

$f'(x) = 10(4x^3) + 5(3x^2) + 20(2x^1) + 15(1) + 40\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + 0 = 40x^3 + 15x^2 + 40x + 15 + 20x^{-\frac{1}{2}}$

i.e., $f'(x) = 40x^3 + 15x^2 + 40x + 15 + 20x^{-\frac{1}{2}}$

6. $\frac{d}{dx}\left[12\sin(x)+8\cos(x)\right]=$

$\frac{d}{dx}\left[12\sin(x)+8\cos(x)\right]=12\left(\cos(x)\right)+8\left(-\sin(x)\right)=12\cos(x)-8\sin(x)$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[12\sin(x)+8\cos(x)\right]=12\cos(x)-8\sin(x)}$$

7. Find the asymptotes and graph: $f(x)=\frac{x^2+1}{x^2+x-6}$

$\boxed{\text{Verticals}}$

1. Find x-values that cause division by zero.

$\Rightarrow x^2+x-6=0$

$\Rightarrow (x+3)(x-2)=0$

$\Rightarrow x=-3$ and $x=2$ are possible vertical asymptotes.

2. Compute the one-sided limits.

$\lim_{x\to -3^-}\frac{x^2+1}{x^2+x-6}=\lim_{x\to -3^-}\frac{x^2+1}{(x+3)(x-2)}=\frac{10}{(-\varepsilon)(-5)}=\frac{\left(\frac{10}{-5}\right)}{(-\varepsilon)}=\frac{(-2)}{-\varepsilon}=+\infty$

$$\boxed{\begin{array}{ll} & x\to -3^- \\ \Rightarrow & x<-3 \\ \Rightarrow & x+3<0 \end{array}}$$

$\lim_{x\to -3^+}\frac{x^2+1}{x^2+x-6}=\lim_{x\to -3^+}\frac{x^2+1}{(x+3)(x-2)}=\frac{10}{(\varepsilon)(-5)}=\frac{\left(\frac{10}{-5}\right)}{(\varepsilon)}=\frac{(-2)}{\varepsilon}=-\infty$

$$\boxed{\begin{array}{ll} & x\to -3^+ \\ \Rightarrow & x>-3 \\ \Rightarrow & x+3>0 \end{array}}$$

Since the one-sided limits are infinite, $x=-3$ *IS a vertical asymptote.*

$\lim_{x\to 2^-}\frac{x^2+1}{x^2+x-6}=\lim_{x\to 2^-}\frac{x^2+1}{(x+3)(x-2)}=\frac{5}{(5)(-\varepsilon)}=\frac{\left(\frac{5}{5}\right)}{(-\varepsilon)}=\frac{1}{-\varepsilon}=-\infty$

$$\boxed{\begin{array}{ll} & x\to 2^- \\ \Rightarrow & x<2 \\ \Rightarrow & x-2<0 \end{array}}$$

$\lim_{x\to 2^+}\frac{x^2+1}{x^2+x-6}=\lim_{x\to 2^+}\frac{x^2+1}{(x+3)(x-2)}=\frac{5}{(5)(+\varepsilon)}=\frac{\left(\frac{5}{5}\right)}{(+\varepsilon)}=\frac{1}{+\varepsilon}=+\infty$

$$\boxed{\begin{array}{ll} & x\to 2^+ \\ \Rightarrow & x>2 \\ \Rightarrow & x-2>0 \end{array}}$$

Since the one-sided limits are infinite, $x=2$ *IS a vertical asymptote.*

$\boxed{\text{Horizontals}}$

Compute the limits as $x \to -\infty$ and as $x \to +\infty$

$\lim_{x\to-\infty} \frac{x^2+1}{x^2+x-6} = \lim_{x\to-\infty} \frac{x^2}{x^2} = \lim_{x\to-\infty} 1 = 1$

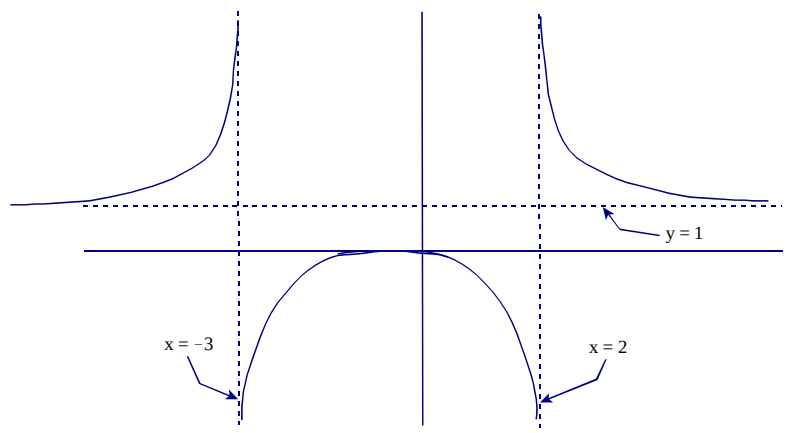
$\lim_{x\to+\infty} \frac{x^2+1}{x^2+x-6} = \lim_{x\to+\infty} \frac{x^2}{x^2} = \lim_{x\to+\infty} 1 = 1$

Since the limits are finite and constant, $y = 1$ is a horizontal asymptote.

Summary:

$\lim_{x\to-3^-} \frac{x^2+1}{x^2+x-6} = +\infty$	
$\lim_{x\to-3^+} \frac{x^2+1}{x^2+x-6} = -\infty$	$\lim_{x\to-\infty} \frac{x^2+1}{x^2+x-6} = 1$
$\lim_{x\to2^-} \frac{x^2+1}{x^2+x-6} = -\infty$	$\lim_{x\to+\infty} \frac{x^2+1}{x^2+x-6} = 1$
$\lim_{x\to2^+} \frac{x^2+1}{x^2+x-6} = +\infty$	

Graph $f(x) = \frac{x^2+1}{x^2+x-6}$



8.

$x =$	$f(x) =$
3.5	-15.1
3.9	-227.8
3.99	-1212.3
3.999	-21156.3
3.9999	-834561.9

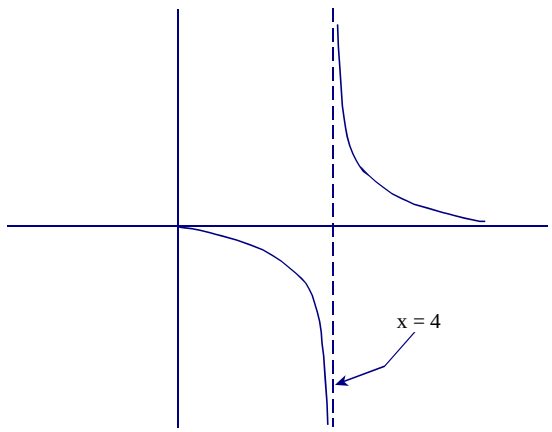
$x =$	$f(x) =$
4.5	15.1
4.1	227.8
4.01	1212.3
4.001	21156.3
4.0001	834561.9

Based on the information in the table above, do the following:

(a) $\lim_{x \to 4^-} f(x) = -\infty$

(b) $\lim_{x \to 4^+} f(x) = \infty$

(c) Graph $f(x)$



9. Compute: $\lim_{x\to 5} \frac{\sqrt{11+x}-4}{x-5} =$

(a) 1. Try Plugging in:

$\lim_{x\to 5} \frac{\sqrt{11+x}-4}{x-5} = \frac{\sqrt{11+(5)}-4}{(5)-5} = \frac{0}{0}$ No Good - Zero Divide!

2. Try Factoring and Cancelling:

$\lim_{x\to 5} \frac{\sqrt{11+x}-4}{x-5} = \lim_{x\to 5} \frac{\sqrt{11+x}-4}{x-5} \cdot \frac{\sqrt{11+x}+4}{\sqrt{11+x}+4} = \lim_{x\to 5} \frac{\left(\sqrt{11+x}\right)^2-(4)^2}{(x-5)\left[\sqrt{11+x}+4\right]}$

$= \lim_{x\to 5} \frac{11+x-16}{(x-5)\left[\sqrt{11+x}+4\right]} = \lim_{x\to 5} \frac{(x-5)}{(x-5)\left[\sqrt{11+x}+4\right]} = \lim_{x\to 5} \frac{1}{\sqrt{11+x}+4} = \frac{1}{\sqrt{11+(5)}+4} =$

$\frac{1}{4+4} = \frac{1}{8}$

$\boxed{\text{i.e., } \lim_{x\to 5} \frac{\sqrt{11+x}-4}{x-5} = \frac{1}{8}}$

10. $f(x) = 2x^2 + 6x + 3$; compute $f'(x)$ **using the definition of derivative.** (i.e., compute $f'(x)$ using the "limiting process.")

$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\left[2(x+\Delta x)^2+6(x+\Delta x)+3\right]-\left(2x^2+6x+3\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\left[2\left(x^2+2x\Delta x+\Delta x^2\right)+6(x+\Delta x)+3\right]-\left(2x^2+6x+3\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\left[\left(2x^2+4x\Delta x+2\Delta x^2\right)+(6x+6\Delta x)+3\right]-\left(2x^2+6x+3\right)}{\Delta x} = \lim_{\Delta x\to 0} \frac{4x\Delta x+2\Delta x^2+6\Delta x}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\Delta x(4x+2\Delta x+6)}{\Delta x} = \lim_{\Delta x\to 0} (4x + 2\Delta x + 6) = 4x + 2(0) + 6 = 4x + 6$

$\boxed{\text{i.e., } f'(x) = 4x + 6}$

EXTRA - WOW! (7 pts) Determine whether or not $f(x)$ is continuous at the point $x = 3$. (Justify your answer.)

11. $f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{for } x < 3 \\ 6 & \text{for } x = 3 \\ 12 - 2x & \text{for } x > 3 \end{cases}$

If $f(x)$ is continuous at the point $x = 3$, then $\lim_{x \to 3} f(x) = f(3)$.

To see if this is true, we'll compute $\lim_{x \to 3} f(x)$.

Since the definiton of $f(x)$ for $x < 3$ is different than the definiton of $f(x)$ for $x > 3$, we must compute the one-sided limits in order to determine whether the limit exists.

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2-9}{x-3} = \lim_{x \to 3^-} \frac{(x+3)(x-3)}{x-3} = \lim_{x \to 3^-} (x+3) = 6$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (12 - 2x) = 12 - 2(3) = 6$

Since the one-sided limits ARE EQUAL, $\lim_{x \to 3} f(x)$ EXISTS and is equal to the common value of the one-sided limits.

i.e., $\lim_{x \to 3} f(x) = 6$

Also, $f(3) = 6$

Therefore, since $\lim_{x \to 3} f(x) = f(3)$,

$f(x)$ IS continuous at $x = 3$