MTH 1126 - Test #1 - Solutions

SPRING 2017

Pat Rossi Name ______________________________

Show CLEARLY how you arrive at your answers

1. Compute: $\int \frac{x^2}{(4x^3-5)^5}\,dx = \int \left(4x^3-5\right)^{-5} x^2\,dx$

1. Is u-sub appropriate?

a. Is there a composite function?

Yes. $\left(4x^3-5\right)^{-5}$

Let $u = \left(4x^3-5\right)$ i.e., "Let $u =$ the 'inner' function"

b. Is there an "approximate function/derivative pair"?

Yes. $\underbrace{\left(4x^3-5\right)}_{\text{function}} \rightarrow \underbrace{x^2}_{\text{deriv}}$

Let $u = \left(4x^3-5\right)$ i.e., "Let $u =$ 'the function'"

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & 4x^3-5 \\ \Rightarrow & \frac{du}{dx} & = & 12x^2 \\ \Rightarrow & du & = & 12x^2dx \\ \Rightarrow & \frac{1}{12}du & = & x^2dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\left(4x^3-5\right)^{-5}}_{u^{-5}} \underbrace{x^2\,dx}_{\frac{1}{12}du} = \int u^{-5}\frac{1}{12}du = \frac{1}{12}\int u^{-5}\,du$$

4. Integrate in terms of u

$$\frac{1}{12}\int u^{-5}\,du = \frac{1}{12}\frac{u^{-4}}{-4} + C = -\frac{1}{48}u^{-4} + C$$

5. Re-write in terms of x

$$\int \frac{x^2}{(4x^3-5)^5}\,dx = \underbrace{-\frac{1}{48}\left(4x^3-5\right)^{-4} + C}_{-\frac{1}{48}u^{-4}+C}$$

2. Compute: $\int \frac{x^2}{(4x^3-5)}\,dx = \int \frac{1}{(4x^3-5)} x^2\,dx$

1. Is u-sub appropriate?

    a. Is there a composite function?

    Yes. $\frac{1}{(4x^3-5)} = \left(4x^3 - 5\right)^{-1}$

    Let $u = \left(4x^3 - 5\right)$ i.e., "Let $u =$ the 'inner' function"

    b. Is there an "approximate function/derivative pair"?

    Yes. $\underbrace{\left(4x^3 - 5\right)}_{\text{function}} \rightarrow \underbrace{x^2}_{\text{deriv}}$

    Let $u = \left(4x^3 - 5\right)$ i.e., "Let $u =$ 'the function'"

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & 4x^3 - 5 \\ \Rightarrow & \frac{du}{dx} & = & 12x^2 \\ \Rightarrow & du & = & 12x^2 dx \\ \Rightarrow & \frac{1}{12} du & = & x^2 dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\frac{1}{(4x^3-5)}}_{\frac{1}{u}} \underbrace{x^2\,dx}_{\frac{1}{12}du} = \int \frac{1}{u}\frac{1}{12}du = \frac{1}{12}\int \frac{1}{u}\,du$$

4. Integrate in terms of u

$$\frac{1}{12}\int \frac{1}{u}\,du = \frac{1}{12}\ln|u| + C$$

5. Re-write in terms of x

$$\int \frac{x^2}{(4x^3-5)}\,dx = \underbrace{\frac{1}{12}\ln\left|4x^3 - 5\right| + C}_{\frac{1}{12}\ln|u|+C}$$

3. Compute: $\frac{d}{dx}\left[e^{\left(4x^5+5x^4\right)}\right] = \underbrace{\dfrac{d}{dx}\left[e^{\left(4x^5+5x^4\right)}\right]}_{\frac{d}{dx}[e^u]} = \underbrace{e^{\left(4x^5+5x^4\right)}}_{e^u} \cdot \underbrace{\left(20x^4+20x^3\right)}_{\frac{du}{dx}} = \left(20x^4+20x^3\right) e^{\left(4x^5+5x^4\right)}$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[e^{\left(4x^5+5x^4\right)}\right] = \left(20x^4+20x^3\right) e^{\left(4x^5+5x^4\right)}}$$

4. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{4x^2-2x}{2x^2+1}}\right)\right] =$

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{4x^2-2x}{2x^2+1}}\right)\right] = \frac{d}{dx}\left[\ln\left(\left(\frac{4x^2-2x}{2x^2+1}\right)^{\frac{1}{2}}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\ln\left(\left(\frac{4x^2-2x}{2x^2+1}\right)\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(4x^2-2x\right) - \ln\left(2x^2+1\right)\right)\right]$$

$$= \frac{1}{2}\left(\underbrace{\frac{1}{4x^2-2x}}_{\frac{i}{u}} \cdot \underbrace{(8x-2)}_{\frac{du}{dx}} - \underbrace{\frac{1}{2x^2+1}}_{\frac{i}{u}} \cdot \underbrace{4x}_{\frac{du}{dx}}\right) = \frac{1}{4x^2-2x}(4x-1) - \frac{1}{2x^2+1}2x = \frac{4x-1}{4x^2-2x} - \frac{2x}{2x^2+1}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[\ln\left(\sqrt{\frac{4x^2-2x}{2x^2+1}}\right)\right] = \frac{4x-1}{4x^2-2x} - \frac{2x}{2x^2+1}}$$

Alternatively:

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{4x^2-2x}{2x^2+1}}\right)\right] = \frac{d}{dx}\left[\underbrace{\ln\left[\left(\frac{4x^2-2x}{2x^2+1}\right)^{\frac{1}{2}}\right]}_{\ln(u)}\right] = \underbrace{\frac{1}{\left(\frac{4x^2-2x}{2x^2+1}\right)^{\frac{1}{2}}}}_{\frac{1}{u}} \cdot \underbrace{\left(\frac{d}{dx}\left[\left(\frac{4x^2-2x}{2x^2+1}\right)^{\frac{1}{2}}\right]\right)}_{\frac{du}{dx}}$$

$$= \left(\frac{2x^2+1}{4x^2-2x}\right)^{\frac{1}{2}} \cdot \frac{1}{2}\left(\frac{4x^2-2x}{2x^2+1}\right)^{-\frac{1}{2}} \underbrace{\frac{(8x-2)\left(2x^2+1\right) - (4x)\left(4x^2-2x\right)}{\left(2x^2+1\right)^2}}_{\text{Quotient Rule}}$$

$$= \left(\frac{2x^2+1}{4x^2-2x}\right)^{\frac{1}{2}} \cdot \frac{1}{2}\left(\frac{2x^2+1}{4x^2-2x}\right)^{\frac{1}{2}} \frac{(8x-2)\left(2x^2+1\right) - (4x)\left(4x^2-2x\right)}{\left(2x^2+1\right)^2}$$

$$= \frac{1}{2}\left(\frac{2x^2+1}{4x^2-2x}\right) \frac{(8x-2)\left(2x^2+1\right) - (4x)\left(4x^2-2x\right)}{\left(2x^2+1\right)^2} = \frac{(4x-1)\left(2x^2+1\right) - (2x)\left(4x^2-2x\right)}{(4x^2-2x)(2x^2+1)}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[\ln\left(\sqrt{\frac{4x^2-2x}{2x^2+1}}\right)\right] = \frac{(4x-1)\left(2x^2+1\right) - (2x)\left(4x^2-2x\right)}{(4x^2-2x)(2x^2+1)}}$$

5. Compute: $\int e^{\left(5x^2+4\right)} x\,dx =$

(a) 1. Is u-sub appropriate?

a. Is there a composite function?

Yes. $e^{\left(5x^2+4\right)}$

Let $u = 5x^2 + 4$

b. Is there an "approximate function/derivative pair"?

Yes. $\underbrace{\left(5x^2+4\right)}_{\text{function}} \to \underbrace{x}_{\text{deriv}}$

Let $u = 5x^2 + 4$

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & 5x^2+4 \\ \Rightarrow & \frac{du}{dx} & = & 10x \\ \Rightarrow & du & = & 10x dx \\ \Rightarrow & \frac{1}{10} du & = & x dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{e^{\left(5x^2+4\right)}}_{e^u} \underbrace{x\,dx}_{\frac{1}{10}du} = \int e^u \frac{1}{10} du = \frac{1}{10} \int e^u \, du$$

4. Integrate in terms of u

$$\frac{1}{10} \int e^u \, du = \frac{1}{10} e^u + C$$

5. Re-write in terms of x

$$\int e^{\left(5x^2+4\right)} x\,dx = \underbrace{\frac{1}{10} e^{\left(5x^2+4\right)} + C}_{\frac{1}{10}e^u + C}$$

6. Compute: $\frac{d}{dx}\left[\text{arcsec}\,(2x)\right] =$

$$\underbrace{\frac{d}{dx}\left[\text{arcsec}\,(2x)\right]}_{\frac{d}{dx}[\text{arcsec}(u)]} = \underbrace{\frac{1}{|2x|\sqrt{(2x)^2-1}}}_{\frac{1}{|u|\sqrt{u^2-1}}} \cdot \underbrace{2}_{\frac{du}{dx}} = \frac{2}{|2x|\sqrt{(2x)^2-1}} = \frac{1}{|x|\sqrt{4x^2-1}}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[\text{arcsec}\,(2x)\right] = \frac{2}{|2x|\sqrt{(2x)^2-1}} = \frac{1}{|x|\sqrt{4x^2-1}}}$$

7. Compute: $\int \frac{1}{\sqrt{16-4x^2}}\,dx \quad = \quad \int \left(16-4x^2\right)^{-\frac{1}{2}}\,dx$

↖ re-write ↗

1 a. Is there a composite function?

Yes. $\left(\underbrace{16-4x^2}_{\text{inner}}\right)^{-\frac{1}{2}}$

Let $u = 16-4x^2$

Is there an “approximate function/derivative pair”?

There does not appear to be an “approximate function/derivative pair.”

Proceeding solely on the strength of Part a, we continue, aware of the possibility that u-substitution might not work.

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & 16-4x^2 \\ \Rightarrow & \frac{du}{dx} & = & -8x \\ \Rightarrow & du & = & -8xdx \\ \Rightarrow & -\frac{1}{8}du & = & xdx \end{array}}$$

3. Analyze in terms of u and du.

$$\underbrace{\int \left(16-4x^2\right)^{-\frac{1}{2}}}_{u^{-\frac{1}{2}}} \cdot \underbrace{dx}_{-\frac{1}{8x}du} =$$

Since we cannot analyze the integral solely in terms of u and du, u-substitution alone will not work in this case.

We must try to get our integral to fit a different form. (See Next Page)

Exercise 7 Continued . . .

$\int \frac{1}{\sqrt{16-4x^2}}\,dx$ compare to: $\int \frac{1}{\sqrt{a^2-u^2}}du$

If this comparison is correct, then:

	$a^2 = 16$
$\Rightarrow$	$a = 4$
	$u^2 = 4x^2 = (2x)^2$
$\Rightarrow$	$u = 2x$
$\Rightarrow$	$\frac{du}{dx} = 2$
$\Rightarrow$	$du = 2dx$
$\Rightarrow$	$\frac{1}{2}du = dx$

Now analyze the integral in terms of u and du.

$$\int \underbrace{\frac{1}{\sqrt{16-4x^2}}}_{\sqrt{a^2-u^2}}\underbrace{dx}_{\frac{1}{2}du} = \int \frac{1}{\sqrt{a^2-u^2}}\frac{1}{2}du = \frac{1}{2}\int \frac{1}{\sqrt{a^2-u^2}}du$$

Integrate:

$$\frac{1}{2}\int \frac{1}{\sqrt{a^2-u^2}}du = \frac{1}{2}\left[\arcsin\left(\frac{u}{a}\right)\right] + C = \frac{1}{2}\left[\arcsin\left(\frac{2x}{4}\right)\right] + C = \frac{1}{2}\arcsin\left(\frac{x}{4}\right) + C$$

i.e., $\int \frac{x}{\sqrt{16-4x^2}}\,dx = \frac{1}{2}\arcsin\left(\frac{x}{4}\right) + C$

8. Compute: $\frac{d}{dx}\left[\arccos\left(x^3\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\arccos\left(x^3\right)\right]}_{\frac{d}{dx}[\arccos(u)]} = \underbrace{-\frac{1}{\sqrt{1-\left(x^3\right)^2}}}_{-\frac{1}{\sqrt{1-u^2}}} \cdot \underbrace{3x^2}_{\frac{du}{dx}} = -\frac{3x^2}{\sqrt{1-x^6}}$$

i.e., $\frac{d}{dx}\left[\arccos\left(x^3\right)\right] = -\frac{3x^2}{\sqrt{1-x^6}}$

9. Compute: $\int \frac{1}{9+7x^4} x\,dx =$

1 a. Is there a composite function?

Yes. $\frac{1}{5+16x^4} = \underbrace{\left(9+7x^4\right)}_{\text{inner}}^{-1}$

Let $u = 9 + 7x^4$

Is there an "approximate function/derivative pair"?

There does not appear to be an "approximate function/derivative pair."

Proceeding solely on the strength of Part a, we continue, aware of the possibility that u-substitution might not work.

2. Compute du

$$\begin{array}{|lrcl|} & u & = & 9+7x^4 \\ \Rightarrow & \frac{du}{dx} & = & 28x^3 \\ \Rightarrow & du & = & 28x^3 dx \\ \Rightarrow & \frac{1}{28} du & = & x^3 dx \end{array}$$

3. Analyze in terms of u and du.

$$\underbrace{\int \frac{1}{9+7x^4}}_{\frac{1}{u}} \cdot \underbrace{x\,dx}_{\frac{1}{28x^2}du} =$$

Since we cannot analyze the integral solely in terms of u and du, u-substitution alone will not work in this case.

We must try to get our integral to fit a different form. (See Next Page)

Exercise 9 Continued . . .

$\int \frac{1}{9+7x^4} x\, dx$ compare to: $\int \frac{1}{a^2+u^2} du$

If this comparison is correct, then:

	$a^2 = 9$
$\Rightarrow$	$a = 3$
	$u^2 = 7x^4 = \left(\sqrt{7}x^2\right)^2$
$\Rightarrow$	$u = \sqrt{7}x^2$
$\Rightarrow$	$\frac{du}{dx} = 2\sqrt{7}x$
$\Rightarrow$	$du = 2\sqrt{7}xdx$
$\Rightarrow$	$\frac{1}{2\sqrt{7}}du = xdx$

Now analyze the integral in terms of u and du.

$$\int \underbrace{\frac{1}{9+7x^4}}_{a^2+u^2} \underbrace{xdx}_{\frac{1}{2\sqrt{7}}du} = \int \frac{1}{a^2+u^2} \frac{1}{2\sqrt{7}} du = \frac{1}{2\sqrt{7}} \int \frac{1}{a^2+u^2} du$$

Integrate:

$$\frac{1}{2\sqrt{7}} \int \frac{1}{a^2+u^2} du = \frac{1}{2\sqrt{7}} \left[\frac{1}{a} \arctan\left(\frac{u}{a}\right)\right] + C = \frac{1}{2\sqrt{7}} \left[\frac{1}{3} \arctan\left(\frac{\sqrt{7}x^2}{3}\right)\right] + C = \frac{1}{6\sqrt{7}} \arctan\left(\frac{\sqrt{7}x^2}{3}\right) + C$$

$$\boxed{\text{i.e., } \int \frac{1}{9+7x^4} x\, dx = \frac{1}{6\sqrt{7}} \arctan\left(\frac{\sqrt{7}x^2}{3}\right) + C}$$

10. $z = \tan(\arccos(4x))$ Re-write this equation as an equivalent algebraic equation.

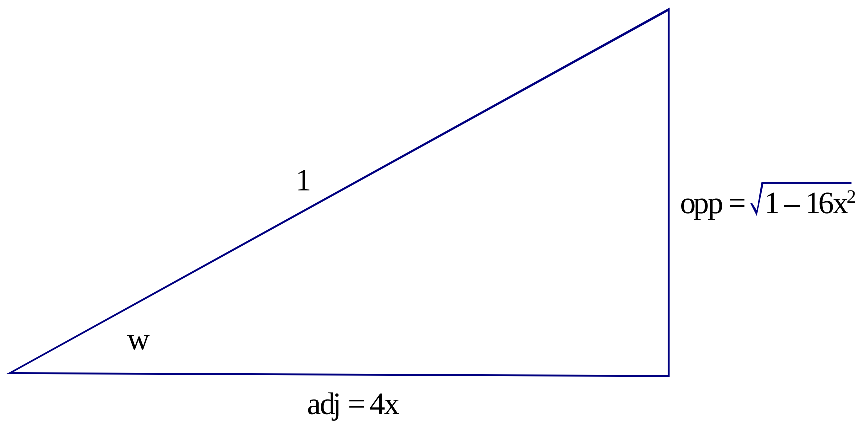
Let $w = \arccos(4x)$

Then "w is the angle whose cosine is $4x$."

i.e., $\cos(w) = 4x$

Draw a right triangle that depicts this relationship.

i.e., $\cos(w) = 4x = \frac{\text{adj}}{\text{hyp}} = \frac{4x}{1}$



We want $z = \tan(\arccos(4x))$

But since $w = \arccos(4x)$,

$\Rightarrow z = \tan(w)$

$\Rightarrow z = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{1-16x^2}}{4x}$

$\boxed{\text{i.e., } z = \frac{\sqrt{1-16x^2}}{4x}}$

Extra: Wow! 10 points (All or nothing)

Compute: $\int \frac{1}{e^{-x}+e^{x}} dx =$

This appears to be of the form $\int \frac{1}{u} du$, where $u = e^{-x} + e^{x}$.

Unfortunately, we don't find du (or a constant multiple of du) in the expression:

$$\int \underbrace{\frac{1}{e^{-x}+e^{x}}}_{\frac{1}{u}} \underbrace{dx}_{\text{NOT a constant Multiple of } du}$$

Therefore, we can't use u-substitution on the integral while it is in the present form.

Looking at the integral $\int \frac{1}{e^{-x}+e^{x}} dx$, I'd like to make it fit the form: $\int \frac{1}{a^2+u^2} du$. But, that would require us to be able to turn the e^{-x} in the denominator into a constant. We can accomplish this (without changing the value of the integrand) by multiplying both numerator and denominator by e^{x}.

$$\int \frac{1}{e^{-x}+e^{x}} dx = \int \frac{e^{x}}{e^{x}} \frac{1}{e^{-x}+e^{x}} dx = \int \frac{e^{x}}{e^{x}(e^{-x}+e^{x})} dx = \int \frac{1}{\left(1+(e^{x})^2\right)} e^{x} dx$$

If we let:

$a = 1$

$u = e^{x}$

$du = e^{x} dx$,

Then this fits (perfectly!) the desired form:

$$\int \underbrace{\frac{1}{\left(1+(e^{x})^2\right)}}_{\frac{1}{a^2+u^2}} \underbrace{e^{x} dx}_{du} = \int \frac{1}{a^2+u^2} du = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C = \frac{1}{1} \arctan\left(\frac{e^{x}}{1}\right) + C = \arctan\left(e^{x}\right) + C$$

i.e., $\int \frac{1}{e^{-x}+e^{x}} dx = \arctan\left(e^{x}\right) + C$