MTH 1125 (2 pm) Test #3 - Solutions

Fall 2023

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Instructions. Show CLEARLY how you arrive at your answers.

1. $f(x) = x^3 - 3x^2 + 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums. (Caution - there are **two** critical numbers. Make sure you get them both!)

   i. Compute $f'(x)$ and find the critical numbers

   $f'(x) = 3x^2 - 6x$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 0$ and solve for x

   $\Rightarrow f'(x) = 3x^2 - 6x = 0$

   $\Rightarrow 3x(x-2) = 0$

   $\Rightarrow 3x = 0 \quad \text{or} \quad (x-2) = 0$

   $\Rightarrow x = 0$ and $x = 2$ are critical numbers.

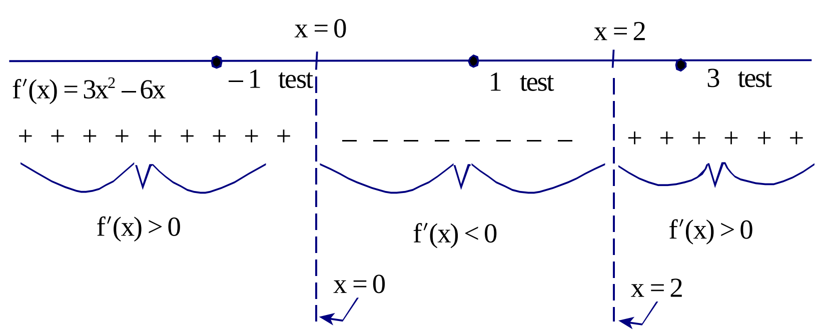
   b. "Type b" ($f'(c)$ is undefined)

   Look for x-value that causes division by zero.

   No "type b" critical numbers

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

3. Pick a "test point" from each interval to plug into $f'(x)$



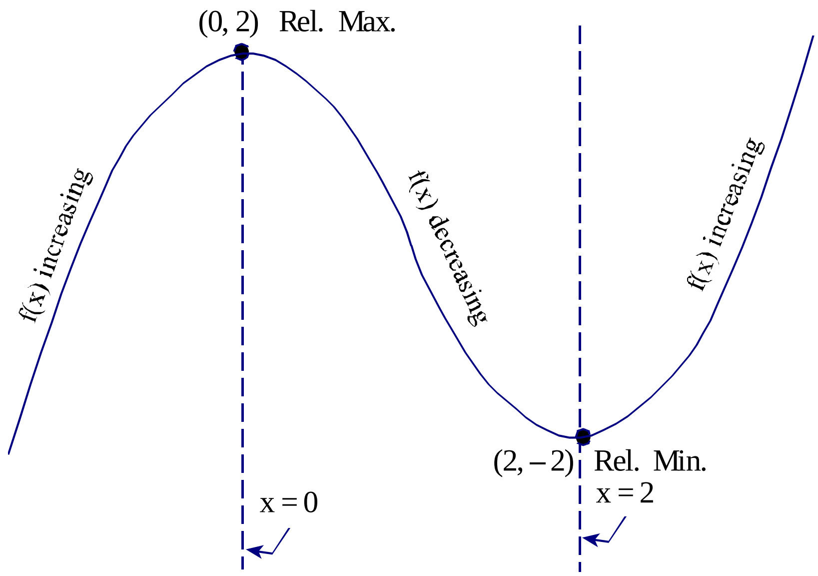
$f(x)$ is **increasing** on the interval(s) $(-\infty, 0)$ and $(2, \infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(0, 2)$

(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(0, f(0)) = (0, 2)$

Rel Min $(2, f(2)) = (2, -2)$

2. $f(x) = \frac{1}{4}x^4 + 2x^3 - \frac{15}{2}x^2 + 6x + 3$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection. Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection. (Caution - there are **two** points of inflection. Make sure you get them both!)

   1. Compute $f''(x)$ and find possible points of inflection.

      $f'(x) = x^3 + 6x^2 - 15x + 6$

      $f''(x) = 3x^2 + 12x - 15$

      Find possible points of inflection:

      a. "Type a" $(f''(x) = 0)$

         Set $f''(x) = 0$

         $\Rightarrow f''(x) = 3x^2 + 12x - 15 = 0$

         $\Rightarrow 3(x^2 + 4x - 5) = 0$

         $\Rightarrow 3(x+5)(x-1) = 0$

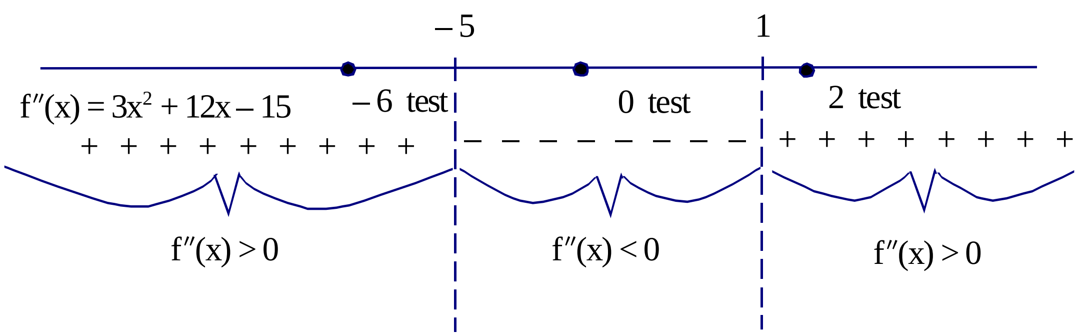
         $x = -5, 1$ possible "type a" points of inflection

      b. "Type b" $(f''(x)$ undefined$)$

         No "Type b" points of inflection

   2. Draw a "sign graph" of $f''(x)$, using the possible points of inflection to partition the x-axis.

   3. Select a test point from each interval and plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-\infty, -5)$ and $(1, \infty)$
(because $f''(x)$ is positive on these intervals)

$f(x)$ is **concave down** on the interval $(-5, 1)$
(because $f''(x)$ is negative on this interval)

Since $f(x)$ changes concavity at $x = -5$ and $x = 1$, the points:
$(-5, f(-5)) = \left(-5, -\frac{1233}{4}\right)$
and
$(1, f(1)) = \left(1, \frac{15}{4}\right)$ **are** points of inflection.

3. $f(x) = 2x^3 + 15x^2 - 84x + 3$ on the interval $[-2, 3]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

Note: [1] $f(x)$ is continuous (since it is a polynomial) on the [2]closed, [3]finite interval $[-2, 3]$. Therefore, we can use the Absolute Max/Min Value Test.

i. Compute $f'(x)$ and find the critical numbers.

$f'(x) = 6x^2 + 30x - 84$

a. "Type a" $(f'(x) = 0)$

$f'(x) = 6x^2 + 30x - 84 = 0$

$\Rightarrow x^2 + 5x - 14 = 0$

$\Rightarrow (x + 7)(x - 2) = 0$

$\Rightarrow x = -7, 2$ are "type a" critical numbers

Since $x = -7$ is not in the interval $[-2, 3]$, we discard it as a critical number.

b. "Type b" ($f'(x)$ is undefined)

No "Type b" critical numbers

ii. Plug endpoints and critical numbers into $f(x)$ (the *original* function)

$f(-2) = 2(-2)^3 + 15(-2)^2 - 84(-2) + 3 = 215$ ← Abs Max Value

$f(2) = 2(2)^3 + 15(2)^2 - 84(2) + 3 = -89$ ← Abs Min Value

$f(3) = 2(3)^3 + 15(3)^2 - 84(3) + 3 = -60$

The Abs Max Value is 215
(attained at $x = -2$)

The Abs Min Value is -89
(attained at $x = 2$)

4. $f(x) = \frac{3}{10}x^{\frac{20}{7}} - x^{\frac{6}{7}} - 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

   1. Compute $f'(x)$ and find the critical numbers

      $f'(x) = \frac{6}{7}x^{\frac{13}{7}} - \frac{6}{7}x^{-\frac{1}{7}} = \frac{6x^{\frac{13}{7}}}{7} - \frac{6}{7x^{\frac{1}{7}}} = \frac{6x^{\frac{13}{7}}}{7}\frac{x^{\frac{1}{7}}}{x^{\frac{1}{7}}} - \frac{6}{7x^{\frac{1}{7}}} = \frac{6x^2-6}{7x^{\frac{1}{7}}}$

      i.e., $f'(x) = \frac{6x^2-6}{7x^{\frac{1}{7}}}$

      a. "Type a" $(f'(c) = 0)$

         Set $f'(x) = 0$ and solve for x

         $\Rightarrow f'(x) = \frac{6x^2-6}{7x^{\frac{1}{7}}} = 0$

         $\Rightarrow 6x^2 - 6 = 0$

         $\Rightarrow x^2 - 1 = 0$

         $\Rightarrow (x+1)(x-1) = 0$

         $\Rightarrow x = -1$ and $x = 1$ are critical numbers.

      b. "Type b" $(f'(c)$ is undefined$)$

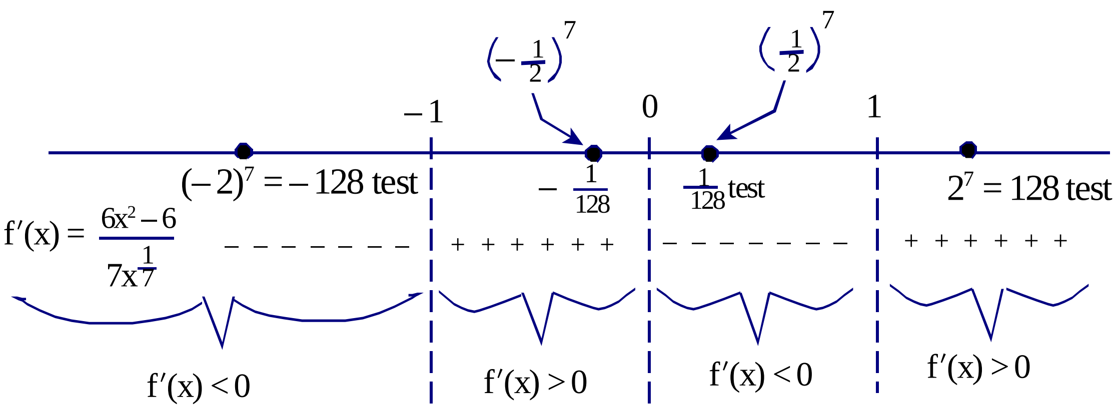
         Look for x-value that causes division by zero.

         $\Rightarrow 7x^{\frac{1}{7}} = 0$

         $\Rightarrow x = 0$ "type b" critical number

   2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

   3. Pick a "test point" from each interval to plug into $f'(x)$



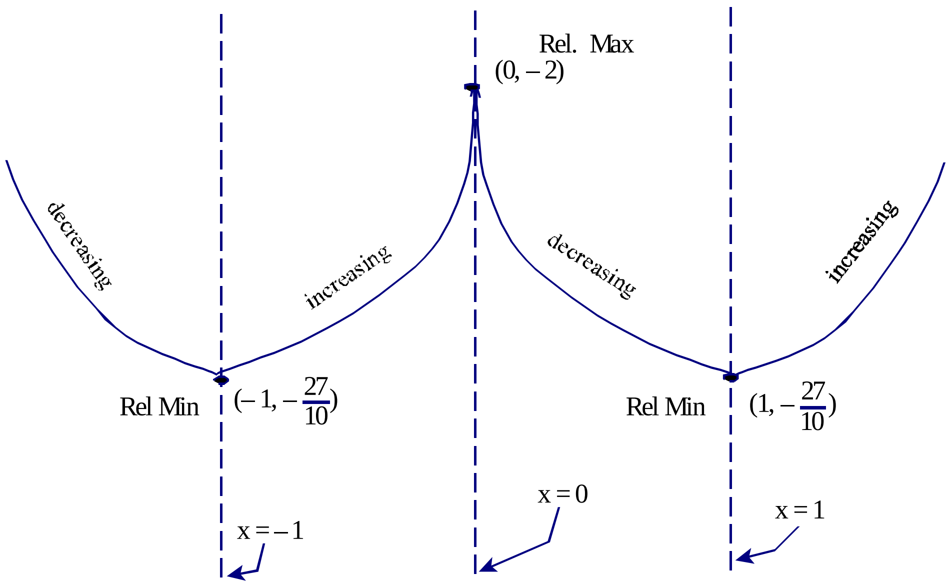
$f(x)$ is **increasing** on the interval(s) $(-1, 0)$ and $(1, \infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-\infty, -1)$ and $(0, 1)$

(because $f'(x)$ is negative on those intervals)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.

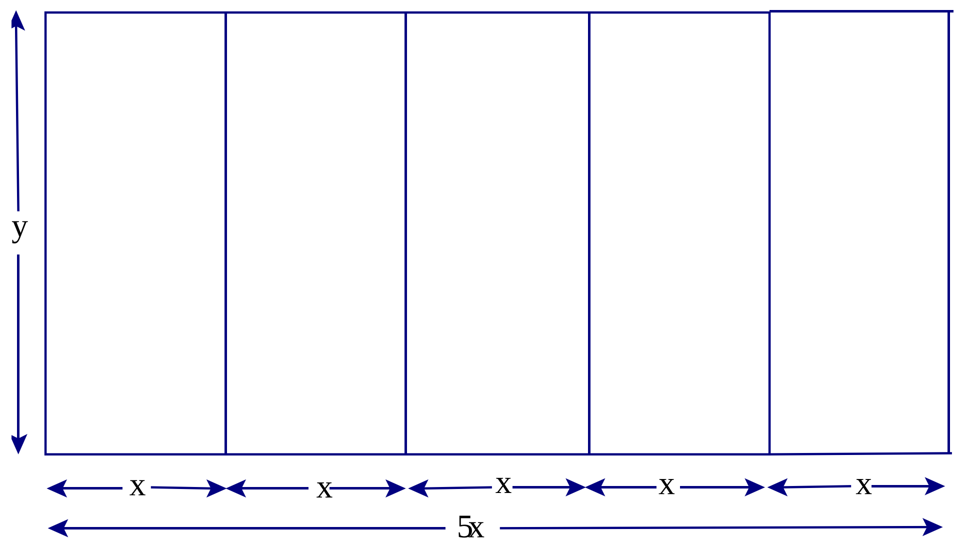


Rel Minimums: $(-1, f(-1)) = \left(-1, -\frac{27}{10}\right)$

and $(1, f(1)) = \left(1, -\frac{27}{10}\right)$

Rel Maximum: $(0, f(0)) = (0, -2)$

5. A rancher has 300 yards of fencing to enclose five adjacent rectangular corrals, as shown below. What overall dimensions should be used so that the enclosed area will be as large as possible?



Solution: Version 1 (Express area A as a function of y)

i. Determine the quantity to be maximized/minimized - give it a name,

   Maximize the overall area of the pen, $A = 5xy$

ii. Express A as a function of *one* variable.

   (Refer to a restriction stated in the problem to do this)

   Restriction: Farmer Joe will use exactly 300 yards of fencing.

   Since the fencing consists of 10 segments of length x and 6 segments of length y, we have:

   $10x + 6y = 300$ yds

   $\Rightarrow 10x = 300$ yds $-6y$

   $\Rightarrow 5x = 150$ yds $-3y$

   Substituting this into the equation $A = 5xy$, we have:

   $A = (150 \text{ yds} - 3y)\, y = 150 \text{ yds } y - 3y^2$

   i.e., $A(y) = 150 \text{ yds } y - 3y^2$

iii. Determine the restrictions on y

$0 \text{ yds} \le y \le \frac{300}{6} \text{ yds}$

i.e., $0 \text{ yds} \le y \le 50 \text{ yds}$

iv. Maximize/minimize, using the techniques of calculus.

Observe: $A(y)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0 \text{ yds}\ ,\ 50 \text{ yds}]$.

Therefore, we can use the Absolute Max/Min Value TestCompute $A'(y)$ and find the critical numbers

$A'(y) = 150 \text{ yds } - 6y$

"Type a" $(A'(c) = 0)$

$A'(y) = 150 \text{ yds } - 6y = 0$

$\Rightarrow 150 \text{ yds } - 6y = 0$

$\Rightarrow 6y = 150 \text{ yds}$

$\Rightarrow y = 25$ yds - critical number

"Type b" $(A'(c)$ is undefined)

There are none.

Plug the critical numbers and endpoints into the *original function.*

$A(0 \text{ yds}) = 150 \text{ yds } (0 \text{ yds}) - 3(0 \text{ yds })^2 = 0 \text{ yds}^2$

$A(25 \text{ yds}) = 150 \text{ yds } (25) - 3(25)^2 = 1875 \text{ yds}^2 \leftarrow$ Abs Max Value

$A(50 \text{ yds}) = 150 \text{ yds } (50 \text{ yds}) - 3(50 \text{ yds})^2 = 0 \text{ yds}^2$

5. Make sure that we've solved the original question (problem).

"What should the overall dimensions ... be"

We have the Abs Max Area when $y = 25$ yds

Length = 5x = $150 \text{ yds } - 3y = 150 \text{ yds } - 3(25 \text{ yds}) = 75$ yds

Width = $y = 25$ yds

Length = 75 yds
Width = 25 yds

Solution: Version 2 (Express area A as a function of x)

i. Determine the quantity to be maximized/minimized - give it a name,

Maximize the overall area of the pen, $A = 5xy$

ii. Express A as a function of *one* variable.

(Refer to a restriction stated in the problem to do this)

Restriction: Farmer Joe will use exactly 300 yards of fencing.

Since the fencing consists of 10 segments of length x and 6 segments of length y, we have:

$10x + 6y = 300$ yds

$\Rightarrow 6y = 300$ yds $-10x$

$\Rightarrow y = 50$ yds $-\frac{5}{3}x$

Substituting this into the equation $A = 5xy$, we have:

$A = 5x\left(50 \text{ yds } - \frac{5}{3}x\right) = 250 \text{ yds } x - \frac{25}{3}x^2$

i.e., $A(x) = 250 \text{ yds } x - \frac{25}{3}x^2$

iii. Determine the restrictions on x

$0 \text{ yds} \le x \le \frac{300}{10} \text{ yds}$

i.e., $0 \text{ yds} \le x \le 30 \text{ yds}$

iv. Maximize/minimize, using the techniques of calculus.

Observe: $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0 \text{ yds}, 30 \text{ yds}]$.

Therefore, we can use the Absolute Max/Min Value TestCompute $A'(x)$ and find the critical numbers

$A'(x) = 250 \text{ yds } -\frac{50}{3}x$

"Type a" $(A'(c) = 0)$

$A'(x) = 250 \text{ yds } -\frac{50}{3}x = 0$

$\Rightarrow 750 \text{ yds } -50x = 0$

$\Rightarrow 50x = 750 \text{ yds}$

$\Rightarrow x = 15$ yds - critical number

"Type b" $(A'(c)$ is undefined)

There are none.

Plug the critical numbers and endpoints into the *original function.*

$A(0\text{ yds}) = 250\text{ yds}\,(0\text{ yds}) - \frac{25}{3}(0\text{ yds})^2 = 0\text{ yds}^2$

$A(15\text{ yds}) = 250\text{ yds}\,(15) - \frac{25}{3}(15)^2 = 1875\text{ yds}^2 \leftarrow$ Abs Max Value

$A(30\text{ yds}) = 250\text{ yds}\,(30\text{ yds}) - \frac{25}{3}(30\text{ yds})^2 = 0\text{ yds}^2$

5. Make sure that we've solved the original question (problem).

"What should the overall dimensions ... be"

We have the Abs Max Area when $x = 15$ yds

Length $= 5\text{x} = (5)(15)$ yds $= 75$ yds

Width $= y = 50\text{ yds} - \frac{5}{3}(15\text{ yds}) = 25$ yds

Length = 75 yds
Width = 25 yds

EXTRA! (Wow! 10 points!)

In the exercise below,
[1]Deterimine the intervals on which $f(x)$ is increasing/decreasing
[2]Identify all relative maximums and minimums
[3]Deterimine the intervals on which $f(x)$ is CCU/CCD
[4]Identify all points of inflections
[5]Graph $f(x)$

$f(x) = x^3 - 3x^2 - 9x + 13$

(Increasing/Decreasing - Max/Mins)

1. Compute $f'(x)$ and find critical numbers

   $f'(x) = 3x^2 - 6x - 9$

   a. "Type a" $(f'(c) = 0)$

      Set $f'(x) = 3x^2 - 6x - 9 = 0$

      $\Rightarrow 3x^2 - 6x - 9 = 0$

      $\Rightarrow x^2 - 2x - 3 = 0$

      $\Rightarrow (x + 1)(x - 3) = 0$

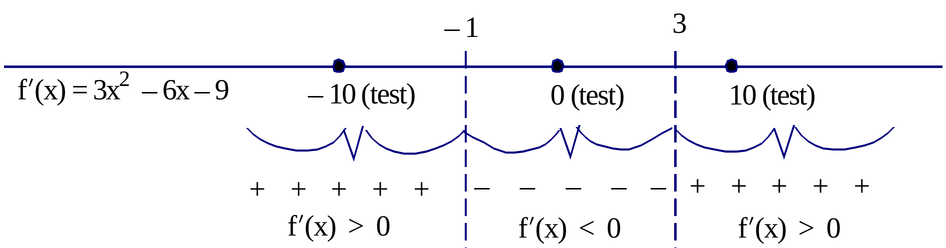
      $\Rightarrow x = -1;\ x = 3$ critical numbers

   b. "Type b" $(f'(c)$ undefined)

      There are none.

2. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

3. From each interval select a "test point" to plug into $f'(x)$



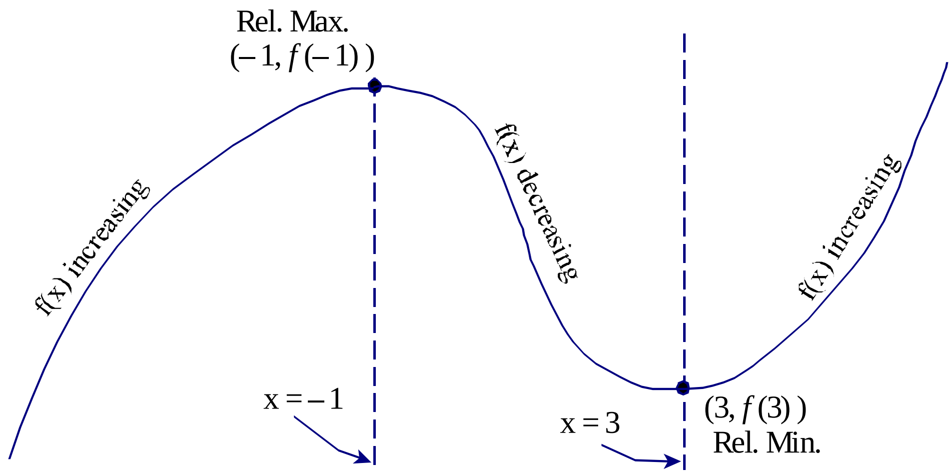
$f(x)$ is **increasing** on the intervals $(-\infty, -1)$ and $(3, \infty)$

(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval $(-1, 3)$

(Because $f'(x)$ is negative on this interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(-1, f(-1)) = (-1, 18)$

Rel Min $(3, f(3)) = (3, -14)$

(Concave Up/Concave Down - Points of inflection)

i. Compute $f''(x)$ and find possible points of inflection

$f'(x) = 3x^2 - 6x - 9$

$f''(x) = 6x - 6$

a. "Type a" ($f''(c) = 0$)

Set $f''(x) = 6x - 6 = 0$

$\Rightarrow 6x - 6 = 0$

$\Rightarrow x - 1 = 0$

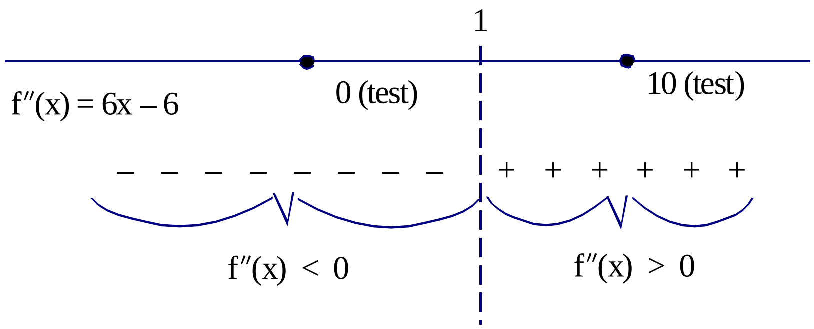
$\Rightarrow x = 1$ possible point of inflection

b. "Type b" ($f''(c)$ undefined)

There are none.

ii. Draw a sign graph of $f''(x)$, using the possible points of inflection to partition the x-axis

iii. From each interval select a "test point" to plug into $f''(x)$



$f(x)$ is **concave down** on the interval $(-\infty, 1)$

(Because $f''(x) < 0$ on these intervals)

$f(x)$ is **concave up** on the interval $(1, \infty)$

(Because $f''(x) > 0$ on this interval)

Since $f(x)$ changes concavity at $x = 1$, the point:

$(1, f(1)) = (1, 2)$ *is* a point of inflection

Graph of $f(x) = 2x^3 - 12x^2 + 18x - 3$