MTH 1126 Practice Test #1_3 - Solutions

Spring 2017

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Instructions

Answers appear on the ANSWERS page. Solutions appear on the SOLUTIONS page.

1. Compute: $\int \left(7x^4 + x^3 + 5x + 10\right) dx =$

$\int \left(7x^4 + x^3 + 5x + 10\right) dx = 7\left[\frac{x^5}{5}\right] + \left[\frac{x^4}{4}\right] + 5\left[\frac{x^2}{2}\right] + 10x + C$

i.e., $\int \left(7x^4 + x^3 + 5x + 10\right) dx = \frac{7}{5}x^5 + \frac{1}{4}x^4 + \frac{5}{2}x^2 + 10x + C$ (Don't forget the "+C")

2. Compute: $\int \left(8\sec(x)\tan(x) + 5\csc^2(x)\right) dx =$

$\int \left(8\sec(x)\tan(x) + 5\csc^2(x)\right) dx = 8\left[\sec(x)\right] + 5\left[-\cot(x)\right] + C$

i.e., $\int \left(8\sec(x)\tan(x) + 5\csc^2(x)\right) dx = 8\sec(x) - 5\cot(x) + C$ (Don't forget the "+C")

3. Compute: $\int_{x=-1}^{x=1} \left(x^3 + 9x^2 + 3\right) dx =$

$$\int_{x=-1}^{x=1} \underbrace{\left(x^3 + 9x^2 + 3\right)}_{f(x)} dx = \underbrace{\left[\frac{1}{4}x^4 + 3x^3 + 3x\right]_{x=-1}^{x=1}}_{F(x)}$$

$$= \underbrace{\left[\frac{1}{4}(1)^4 + 3(1)^3 + 3(1)\right]}_{F(1)} - \underbrace{\left[\frac{1}{4}(-1)^4 + 3(-1)^3 + 3(-1)\right]}_{F(-1)} = 12$$

i.e., $\int_{x=-1}^{x=1} \left(x^3 + 9x^2 + 3\right) dx = 12$

4. Compute: $\int \sqrt{4x^3+6x}\left(6x^2+3\right)dx \underbrace{=}_{\text{Re-write}} \int \left(4x^3+6x\right)^{\frac{1}{2}}\left(6x^2+3\right)dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(4x^3+6x\right)^{\frac{1}{2}}$ (A function raised to a power is always a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(4x^3+6x\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(4x^3+6x\right)}_{\text{function}} - - - - - \rightarrow \underbrace{\left(6x^2+3\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(4x^3+6x\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 4x^3+6x \\ \Rightarrow \frac{du}{dx} &= 12x^2+6 \\ \Rightarrow du &= \left(12x^2+6\right)dx \\ \Rightarrow \frac{1}{2}du &= \left(6x^2+3\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(4x^3+6x\right)^{\frac{1}{2}}}_{u^{\frac{1}{2}}}\underbrace{\left(6x^2+3\right)dx}_{\frac{1}{2}du} = \int u^{\frac{1}{2}} \cdot \frac{1}{2}du = \frac{1}{2}\int u^{\frac{1}{2}}du$$

4. Integrate (in terms of u).

$$\frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \frac{1}{3}u^{\frac{3}{2}} + C$$

5. Re-express in terms of the original variable, x.

$$\int \sqrt{4x^3+6x}\left(6x^2+3\right)dx = \underbrace{\frac{1}{3}\left(4x^3+6x\right)^{\frac{3}{2}} + C}_{\frac{1}{3}u^{\frac{3}{2}}+C}$$

i.e., $\int \sqrt{4x^3+6x}\left(6x^2+3\right)dx = \frac{1}{3}\left(4x^3+6x\right)^{\frac{3}{2}} + C$

5. Compute: $\int \sec\left(x^2\right)\tan\left(x^2\right)\, x\, dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\sec\left(x^2\right)\tan\left(x^2\right)$ ↖ ↗ inner — the "outer" is sec ( ) tan ( )

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(x^2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(x^2\right)}_{\text{function}} - - - - \rightarrow \underbrace{(x)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(x^2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= x^2 \\ \Rightarrow \frac{du}{dx} &= 2x \\ \Rightarrow du &= 2xdx \\ \Rightarrow \frac{1}{2}du &= xdx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\sec\left(x^2\right)\tan\left(x^2\right)}_{\sec(u)\tan(u)} \underbrace{xdx}_{\frac{1}{2}du} = \int \sec(u)\tan(u)\ \frac{1}{2}du = \frac{1}{2}\int \sec(u)\tan(u)\, du$$

4. Integrate (in terms of u).

$$\frac{1}{2}\int \sec(u)\tan(u)\, du = \frac{1}{2}\left[\sec(u)\right] + C = \frac{1}{2}\sec(u) + C$$

5. Re-express in terms of the original variable, x.

$$\int \sec\left(x^2\right)\tan\left(x^2\right)\, x\, dx = \underbrace{\frac{1}{2}\sec\left(x^2\right) + C}_{\frac{1}{2}\sec(u)+C}$$

$$\text{i.e., } \int \sec\left(x^2\right)\tan\left(x^2\right)\, x\, dx = \frac{1}{2}\sec\left(x^2\right) + C$$

6. Compute: $\int \frac{3x^2+x+2}{2x^3+x^2+4x}\,dx \underbrace{=}_{\text{re-write}} \int \frac{1}{2x^3+x^2+4x}\left(3x^2+x+2\right)dx$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{2x^3+x^2+4x}$ is the same as $\left(2x^3+x^2+4x\right)^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(2x^3+x^2+4x\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(2x^3+x^2+4x\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(3x^2+x+2\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(2x^3+x^2+4x\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 2x^3+x^2+4x \\ \Rightarrow \frac{du}{dx} &= 6x^2+2x+4 \\ \Rightarrow du &= \left(6x^2+2x+4\right)dx \\ \Rightarrow \frac{1}{2}du &= \left(3x^2+x+2\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\frac{1}{2x^3+x^2+4x}}_{\frac{1}{u}} \underbrace{\left(3x^2+x+2\right)dx}_{\frac{1}{2}du} = \int \frac{1}{u}\cdot\frac{1}{2}du = \frac{1}{2}\int \frac{1}{u}du$$

4. Integrate (in terms of u).

$\frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\left[\ln|u|\right] + C = \frac{1}{2}\ln|u| + C$

5. Re-express in terms of the original variable, x.

$$\int \frac{3x^2+x+2}{2x^3+x^2+4x}dx = \underbrace{\frac{1}{2}\ln\left|2x^3+x^2+4x\right| + C}_{\frac{1}{2}\ln|u|+C}$$

i.e., $\int \frac{3x^2+x+2}{2x^3+x^2+4x}dx = \frac{1}{2}\ln\left|2x^3+x^2+4x\right| + C$

7. Compute: $\frac{d}{dx}\left[\ln\left(\tan\left(x\right)\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(\tan\left(x\right)\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{\tan\left(x\right)}}_{\frac{1}{g(x)}} \cdot \underbrace{\sec^2\left(x\right)}_{g'(x)} = \frac{\sec^2(x)}{\tan(x)}$$

i.e., $\frac{d}{dx}\left[\ln\left(\tan\left(x\right)\right)\right] = \frac{\sec^2(x)}{\tan(x)}$

8. Compute: $\frac{d}{dx}\left[\ln\left(8x^3 + 5x\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(8x^3 + 5x\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{8x^3 + 5x}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(24x^2 + 5\right)}_{g'(x)} = \frac{24x^2+5}{8x^3+5x}$$

i.e., $\frac{d}{dx}\left[\ln\left(8x^3 + 5x\right)\right] = \frac{24x^2+5}{8x^3+5x}$

9. Compute: $\frac{d}{dx}\left[\ln\left(x\sin\left(x\right)\right)\right] =$

Remark: We can compute this derivative directly, in it's current form, but it would be much easier to use the *algebraic properties* of natural logarithms to simplify the expression first.

$$\frac{d}{dx}\left[\ln\left(x\sin\left(x\right)\right)\right] = \underbrace{\frac{d}{dx}\left[\ln\left(x\right) + \ln\left(\sin\left(x\right)\right)\right]}_{\ln(ab) = \ln(a) + \ln(b)}$$

NOW we're ready to compute the derivative!

$$\frac{d}{dx}\left[\ln\left(x\sin\left(x\right)\right)\right] = \frac{d}{dx}\left[\ln\left(x\right) + \ln\left(\sin\left(x\right)\right)\right] = \left[\frac{1}{x} + \frac{1}{\sin(x)}\cos\left(x\right)\right] = \frac{1}{x} + \cot\left(x\right)$$

i.e., $\frac{d}{dx}\left[\ln\left(x\sin\left(x\right)\right)\right] = \frac{1}{x} + \cot\left(x\right)$

10. Compute: $\int_{x=0}^{x=1} \left(1-x^2\right)^2 x\,dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(1-x^2\right)^2$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(1-x^2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(1-x^2\right)}_{\text{function}} - - - - - \rightarrow \underbrace{x}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(1-x^2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 1-x^2 \\ \Rightarrow \frac{du}{dx} &= -2x \\ \Rightarrow du &= -2xdx \\ \Rightarrow -\tfrac{1}{2}du &= xdx \end{aligned}$$

When $x=0$, $u = 1-x^2 = 1-(0)^2 = 1$
When $x=1$, $u = 1-x^2 = 1-(1)^2 = 0$

3. Analyze in terms of u and du

$$\int_{x=0}^{x=1} \underbrace{\left(1-x^2\right)^2}_{u^2} \underbrace{xdx}_{-\frac{1}{2}du} = \int_{u=1}^{u=0} u^2 \cdot \left(-\tfrac{1}{2}\right) du = -\tfrac{1}{2}\int_{u=1}^{u=0} u^2 du$$

Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$-\tfrac{1}{2}\int_{u=1}^{u=0} u^2 du = -\tfrac{1}{2}\left[\tfrac{u^3}{3}\right]_{u=1}^{u=0} = -\tfrac{1}{6}\left[u^3\right]_{u=1}^{u=0} = \underbrace{-\frac{1}{6}(0)^3}_{F(0)} - \underbrace{\left(-\frac{1}{6}(1)^3\right)}_{F(1)} = 0 - \left(-\tfrac{1}{6}\right) = \tfrac{1}{6}$$

i.e., $\int_{x=0}^{x=1} \left(1-x^2\right)^2 x\,dx = \frac{1}{6}$

11. Write the given equation in algebraic form:

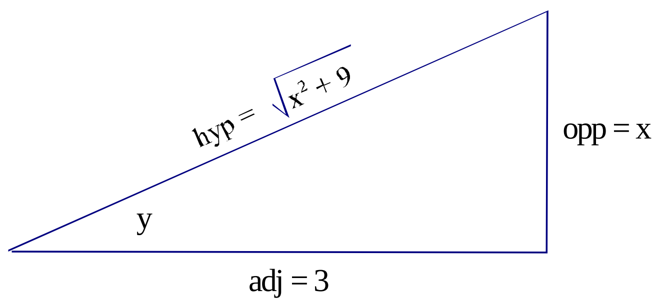
$z = \cos\left(\arctan\left(\frac{x}{3}\right)\right)$

Let $y = \arctan\left(\frac{x}{3}\right)$

This is the same as saying "y is the angle whose tangent is $\frac{x}{3}$."

i.e., $\tan(y) = \frac{x}{3}$.

Let's draw a triangle to depict this relationship.



By Pythagorean's Theorem, $(opp)^2 + (adj)^2 = (hyp)^2$

$\Rightarrow (hyp) = \sqrt{(opp)^2 + (adj)^2}$

$\Rightarrow hyp = \sqrt{x^2 + (3)^2} = \sqrt{x^2+9}.$

Recall: we want $z = \cos\left(\arctan\left(\frac{x}{3}\right)\right) = \cos(y)$

From the picture, $z = \cos(y) = \frac{adj}{hyp} = \frac{3}{\sqrt{x^2+9}}$

Hence, $z = \cos\left(\arctan\left(\frac{x}{3}\right)\right) = \frac{3}{\sqrt{x^2+9}}$

i.e., $z = \frac{3}{\sqrt{x^2+9}}$

12. Compute: $\frac{d}{dx}\left[\ln\left(e^{x^2}\cdot\sin x\right)\right] =$

$\frac{d}{dx}\left[\ln\left(e^{x^2}\cdot\sin x\right)\right] = \frac{d}{dx}\left[\ln\left(e^{x^2}\right) + \ln(\sin(x))\right]$

$= \frac{d}{dx}\left[x^2 + \underbrace{\ln(\sin(x))}_{\ln(u)}\right] = 2x + \underbrace{\frac{1}{\sin x}}_{\frac{1}{u}} \cdot \underbrace{\cos(x)}_{\frac{du}{dx}} = 2x + \cot(x)$

i.e., $\frac{d}{dx}\left[\ln\left(e^{x^2}\cdot\sin x\right)\right] = 2x + \cot(x)$

Alternate Solution (see next page):

Alternatively:

$$\underbrace{\frac{d}{dx}\left[\ln\left(e^{x^2}\sin(x)\right)\right]}_{\frac{d}{dx}[\ln(u)]} = \underbrace{\frac{1}{e^{x^2}\sin(x)}}_{\frac{1}{u}} \cdot \underbrace{\left(\left(e^{x^2}\cdot 2x\right)\cdot(\sin(x)) + \cos(x)\cdot e^{x^2}\right)}_{\frac{du}{dx} \text{ using the product rule}}$$

$$= \frac{1}{e^{x^2}\sin(x)} \cdot e^{x^2}\left(2x\cdot\sin(x)+\cos(x)\right) = \frac{2x\ \sin(x)+\cos(x)}{\sin(x)} = 2x + \cot x$$

$$\text{i.e., } \frac{d}{dx}\left[\ln\left(e^{x^2}\cdot\sin x\right)\right] = 2x + \cot(x)$$

13. Compute: $\int \frac{e^x}{7+e^{2x}}dx \quad = \quad \int \frac{1}{\left(\sqrt{7}\right)^2+(e^x)^2}\cdot e^x\,dx$

↖ re-write ↗

$\frac{1}{\left(\sqrt{7}\right)^2+(e^x)^2}\cdot e^x\,dx$ appears to fit the form: $\int \frac{1}{a^2+u^2}du$

If this analysis is correct, then:

	a^2	$=$	7
$\Rightarrow$	a	$=$	$\sqrt{7}$
$\Rightarrow$	u^2	$=$	e^{2x}
$\Rightarrow$	u	$=$	e^x
$\Rightarrow$	$\frac{du}{dx}$	$=$	e^x
$\Rightarrow$	du	$=$	$e^x dx$

Now analyze the integral in terms of u and du.

$$\int \frac{e^x}{7+e^{2x}}dx = \int \underbrace{\frac{1}{\left(\sqrt{7}\right)^2+(e^x)^2}}_{\frac{1}{a^2+u^2}}\underbrace{e^x dx}_{du} = \int \frac{1}{a^2+u^2}du$$

Integrate:

$$\int \frac{1}{a^2+u^2}du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C = \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{e^x}{\sqrt{7}}\right) + C$$

$$\text{i.e., } \int \frac{e^x}{7+e^{2x}}dx = \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{e^x}{\sqrt{7}}\right) + C$$

14. Compute: $\frac{d}{dx}\left[e^{\tan\left(4x^2\right)}\right] =$

$$\frac{d}{dx}\left[\underbrace{e^{\tan\left(4x^2\right)}}_{e^u}\right] = \underbrace{e^{\tan\left(4x^2\right)}}_{e^u}\cdot\underbrace{\sec^2\left(4x^2\right)\cdot 8x}_{\frac{du}{dx}} = 8x\sec^2\left(4x^2\right)e^{\tan\left(4x^2\right)}$$

$$\text{i.e., } \frac{d}{dx}\left[e^{\tan\left(4x^2\right)}\right] = 8x\sec^2\left(4x^2\right)e^{\tan\left(4x^2\right)}$$

15. Compute: $\int e^{\sec(4x)} \sec(4x) \tan(4x)\, dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $e^{\sec(4x)}$

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \sec(4x)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\sec(4x)}_{\text{function}} - - - - \rightarrow \underbrace{\sec(4x)\tan(4x)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \sec(4x)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= \sec(4x) \\ \Rightarrow \frac{du}{dx} &= 4\sec(4x)\tan(4x) \\ \Rightarrow du &= 4\sec(4x)\tan(4x)\,dx \\ \Rightarrow \frac{1}{4}du &= \sec(4x)\tan(4x)\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{e^{\sec(4x)}}_{e^u} \underbrace{\sec(4x)\tan(4x)\,dx}_{\frac{1}{4}du} = \int e^u \tfrac{1}{4} du = \tfrac{1}{4} \int e^u du$$

4. Integrate (in terms of u).

$$\tfrac{1}{4} \int e^u du = \tfrac{1}{4} e^u + C = \tfrac{1}{4} e^{\sec(4x)} + C$$

5. Re-express in terms of the original variable, x.

$$\int e^{\sec(4x)} \sec(4x)\tan(4x)\,dx = \underbrace{\frac{1}{4} e^{\sec(4x)} + C}_{\frac{1}{4}e^u + C}$$

i.e., $\int e^{\sec(4x)} \sec(4x)\tan(4x)\,dx = \frac{1}{4} e^{\sec(4x)} + C$

16. Compute: $\int_{x=1}^{x=3} \frac{e^{\frac{3}{x}}}{x^2}\,dx \quad = \quad \int e^{3x^{-1}} \cdot x^{-2}\,dx$

re-write

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $e^{3x^{-1}}$

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = 3x^{-1}$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{3x^{-1}}_{\text{function}} - - - - \rightarrow \underbrace{x^{-2}\,dx}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = 3x^{-1}$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 3x^{-1} \\ \Rightarrow \frac{du}{dx} &= -3x^{-2} \\ \Rightarrow du &= -3x^{-2}dx \\ \Rightarrow -\tfrac{1}{3}du &= x^{-2}dx \end{aligned}$$
when $x = 1;\ u = 3\,(1)^{-1} = 3$
when $x = 3;\ u = 3\,(3)^{-1} = 1$

3. Analyze in terms of u and du

$$\int_{x=1}^{x=3} \underbrace{e^{3x^{-1}}}_{e^u} \underbrace{x^{-2}dx}_{-\frac{1}{3}du} = \int_{u=3}^{u=1} e^u \cdot \left(-\tfrac{1}{3}du\right) = -\tfrac{1}{3}\int_{u=3}^{u=1} e^u du$$

4. Integrate (in terms of u).

$$-\tfrac{1}{3}\int_{u=3}^{u=1} e^u du = -\tfrac{1}{3}\left[e^u\right]_{u=3}^{u=1} = -\tfrac{1}{3}\left[e^1 - e^3\right] = \tfrac{1}{3}\left(e^3 - e\right)$$

i.e., $\int_{x=1}^{x=3} \frac{e^{\frac{3}{x}}}{x^2}\,dx = \frac{1}{3}\left(e^3 - e\right)$

17. Compute: $\frac{d}{dx}\left[e^{(\sin(x)+\cos(x))}\right] =$

$$\boxed{\begin{array}{c} u = \sin(x) + \cos(x) \\ \hline \frac{du}{dx} = \cos(x) - \sin(x) \end{array}}$$

$$\underbrace{\frac{d}{dx}\left[e^{(\sin(x)+\cos(x))}\right]}_{\frac{d}{dx}[e^u]} = \underbrace{e^{(\sin(x)+\cos(x))}}_{e^u} \cdot \underbrace{(\cos(x) - \sin(x))}_{\frac{du}{dx}} = e^{(\sin(x)+\cos(x))} \cdot (\cos(x) - \sin(x))$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[e^{(\sin(x)+\cos(x))}\right] = e^{(\sin(x)+\cos(x))} \cdot (\cos(x) - \sin(x))}$$

18. Write the given equation in algebraic form:

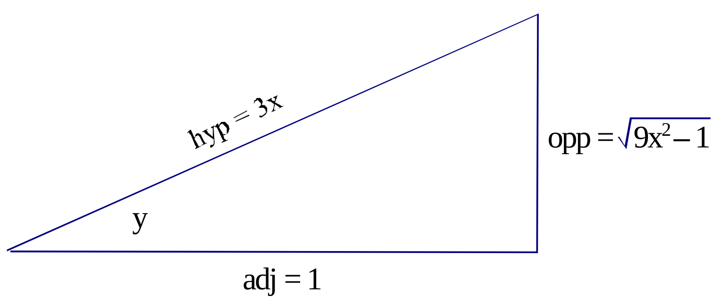
$z = \tan(\text{arcsec}(3x))$

Let $y = \text{arcsec}(3x)$

This is the same as saying "y is the angle whose secant is $3x$."

i.e., $\sec(y) = 3x$.

Let's draw a triangle to depict this relationship.



By Pythagorean's Theorem, $(opp)^2 + (adj)^2 = (hyp)^2$

$\Rightarrow (opp)^2 = (hyp)^2 - (adj)^2$

$\Rightarrow opp = \sqrt{(hyp)^2 - (adj)^2}$

$\Rightarrow opp = \sqrt{(3x)^2 - 1^2} = \sqrt{9x^2 - 1}$.

Recall: we want $z = \tan(\text{arcsec}(3x)) = \tan(y)$

From the picture, $z = \tan(y) = \frac{opp}{adj} = \sqrt{9x^2 - 1}$

Hence, $z = \tan(\text{arcsec}(3x)) = \sqrt{9x^2 - 1}$

$\boxed{\text{i.e., } z = \sqrt{9x^2 - 1}}$

19. Compute: $\int e^{\left(4x^2-3x\right)}\,(16x-6)\;dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

      Yes! $\quad e^{\left(4x^2-3x\right)}$

      Let $u =$ the "inner" of the composite function

      $\Rightarrow u = \left(4x^2-3x\right)$

   b. Is there an (approximate) function/derivative pair?

      Yes! $\quad \underbrace{\left(4x^2-3x\right)}_{\text{function}} - - - - - \rightarrow \underbrace{(16x-6)}_{\text{deriv}}$

      Let $u =$ the "function" of the function/deriv pair

      $\Rightarrow u = \sec(4x)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

      (i.e., do criteria **a** and **b** suggest the same choice of u ?)

      Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{array}{rcl} u & = & 4x^2-3x \\ \Rightarrow \frac{du}{dx} & = & (8x-3) \\ \Rightarrow du & = & (8x-3)\,dx \\ \Rightarrow 2du & = & (16x-6)\,dx \end{array}$$

3. Analyze in terms of u and du

$$\int \underbrace{e^{\left(4x^2-3x\right)}}_{e^u}\underbrace{(16x-6)dx}_{2du} = \int e^u \cdot 2du = 2\int e^u du$$

4. Integrate (in terms of u).

   $2\int e^u du = 2e^u + C$

5. Re-express in terms of the original variable, x.

$$\int e^{\left(4x^2-3x\right)}\,(16x-6)\;dx = \underbrace{2e^{\left(4x^2-3x\right)}+C}_{2e^u+C}$$

i.e., $\int e^{\left(4x^2-3x\right)}\,(16x-6)\;dx = 2e^{\left(4x^2-3x\right)}+C$

20. Compute: $\int \frac{\sec^2(x)}{\tan(x)\sqrt{\tan^2(x)-9}}dx \quad = \quad \int \frac{1}{\tan(x)\sqrt{\tan^2(x)-(3)^2}} \cdot \sec^2(x)\ dx$

$\nwarrow \quad \nearrow$

re-write

$\int \frac{1}{\tan(x)\sqrt{\tan^2(x)-(3)^2}} \cdot \sec^2(x)\ dx$ appears to fit the form: $\int \frac{1}{u\sqrt{u^2-a^2}}du$

If this analysis is correct, then:

$$\boxed{\begin{array}{llll} & a^2 & = & 9 \\ \Rightarrow & a & = & 3 \\ \Rightarrow & u^2 & = & \tan^2(x) \\ \Rightarrow & u & = & \tan(x) \\ \Rightarrow & \frac{du}{dx} & = & \sec^2(x) \\ \Rightarrow & du & = & \sec^2(x)\,dx \end{array}}$$

Now analyze the integral in terms of u and du.

$$\int \frac{\sec^2(x)}{\tan(x)\sqrt{\tan^2(x)-9}}dx = \int \underbrace{\frac{1}{\tan(x)\sqrt{\tan^2(x)-(3)^2}}}_{\frac{1}{u\sqrt{u^2-a^2}}} \cdot \underbrace{\sec^2(x)\ dx}_{du} = \int \frac{1}{u\sqrt{u^2-a^2}}du$$

Integrate:

$$\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C = \frac{1}{3}\sec^{-1}\left(\frac{|\tan(x)|}{3}\right) + C$$

$$\boxed{\text{i.e., } \int \frac{\sec^2(x)}{\tan(x)\sqrt{\tan^2(x)-9}}dx = \frac{1}{3}\sec^{-1}\left(\frac{|\tan(x)|}{3}\right) + C}$$

21. Compute:

$$\int \frac{x^3+1}{(x^4+4x)^2}dx \underset{\nwarrow \text{ re-write } \nearrow}{=} \int \frac{1}{(x^4+4x)^2}\left(x^3+1\right)dx \underset{\nwarrow \text{ re-write } \nearrow}{=} \int \left(x^4+4x\right)^{-2}\left(x^3+1\right)dx$$

(a) 1. Is u-sub appropriate?

a. Is there a composite function?

Yes. $\left(x^4+4x\right)^{-2}$

Let $u = x^4+4x$

b. Is there an "approximate function/derivative pair"?

Yes. $\left(x^4+4x\right) \to \left(x^3+1\right)$

Let $u = x^4+4x$

2. Compute du

$$\boxed{\begin{array}{l} u = x^4+4x \\ du = \left(4x^3+4\right)dx \\ \frac{1}{4}du = \left(x^3+1\right)dx \end{array}}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(x^4+4x\right)^{-2}}_{u^{-2}}\underbrace{\left(x^3+1\right)dx}_{\frac{1}{4}du} = \int u^{-2}\left(\tfrac{1}{4}du\right) = \tfrac{1}{4}\int u^{-2}du$$

4. Integrate in terms of u

$$\tfrac{1}{4}\int u^{-2}du = \tfrac{1}{4}\left[\frac{u^{-1}}{(-1)}\right] + C = -\tfrac{1}{4}u^{-1} + C$$

5. Re-write in terms of x

$$\int \frac{x^3+1}{(x^4+4x)^2}dx = \underbrace{-\frac{1}{4}\left(x^4+4x\right)^{-1} + C}_{-\frac{1}{4}u^{-1}+C}$$

$$\boxed{\int \frac{x^3+1}{(x^4+4x)^2}dx = -\tfrac{1}{4}\left(x^4+4x\right)^{-1} + C}$$

22. Compute: $\int \frac{x^3+1}{x^4+4x} dx \quad = \quad \int \frac{1}{x^4+4x}\left(x^3+1\right)\, dx$

↖ re-write ↗

(a) 1. Is u-sub appropriate?

a. Is there a composite function?

Yes. $\frac{1}{x^4+4x} = \left(x^4+4x\right)^{-1}$

Let $u = x^4+4x$

b. Is there an "approximate function/derivative pair"?

Yes. $\left(x^4+4x\right) \rightarrow \left(x^3+1\right)$

Let $u = x^4+4x$

2. Compute du

$$\boxed{\begin{array}{l} u = x^4+4x \\ du = \left(4x^3+4\right) dx \\ \frac{1}{4}du = \left(x^3+1\right) dx \end{array}}$$

3. Analyze in terms of u and du

$$\int \underbrace{\frac{1}{x^4+4x}}_{\frac{1}{u}} \underbrace{\left(x^3+1\right)}_{\frac{1}{4}du} dx = \int \frac{1}{u}\left(\frac{1}{4}du\right) = \frac{1}{4}\int \frac{1}{u} du$$

4. Integrate in terms of u

$\frac{1}{4}\int \frac{1}{u}du = \frac{1}{4}\left[\ln|u|\right] + C = \frac{1}{4}\ln|u| + C$

5. Re-write in terms of x

$$\int \frac{x^3+1}{x^4+4x}dx = \underbrace{\frac{1}{4}\ln\left|x^4+4x\right| + C}_{\frac{1}{4}\ln|u|+C}$$

$$\boxed{\int \frac{x^3+1}{x^4+4x}dx = \frac{1}{4}\ln\left|x^4+4x\right| + C}$$

23. Compute: $\frac{d}{dx}\left[e^{\sin(x)\cos(x)}\right] =$

$$\underbrace{\frac{d}{dx}\left[e^{\sin(x)\cos(x)}\right]}_{\frac{d}{dx}[e^u]} = \underbrace{e^{\sin(x)\cos(x)}}_{e^u} \cdot \underbrace{\left(\cos(x)\cos(x) + (-\sin(x))\sin(x)\right)}_{\frac{du}{dx}} = \left(\cos^2(x) - \sin^2(x)\right) e^{\sin(x)\cos(x)}$$

$$= \cos(2x)\, e^{\sin(x)\cos(x)}$$

$\boxed{\text{i.e., } \frac{d}{dx}\left[e^{\sin(x)\cos(x)}\right] = \cos(2x)\, e^{\sin(x)\cos(x)}}$

24. Compute: $\frac{d}{dx}\left[\sec^{-1}(e^x)\right] = \underbrace{\frac{d}{dx}\left[\operatorname{arcsec}(e^x)\right]}_{\frac{d}{dx}[\operatorname{arcsec}(u)]} = \underbrace{\frac{1}{|e^x|\sqrt{(e^x)^2 - 1}}}_{\frac{1}{|u|\sqrt{u^2-1}}} \cdot \underbrace{e^x}_{\frac{du}{dx}} = \frac{1}{e^x\sqrt{e^{2x}-1}} e^x = \frac{1}{\sqrt{e^{2x}-1}}$

$\boxed{\frac{d}{dx}\left[\sec^{-1}(e^x)\right] = \frac{1}{\sqrt{e^{2x}-1}}}$

25. Given that $\ln(3) \approx 1.1$ and $\ln(6) \approx 1.8$, approximate the following:

(a) $\ln(2) =$

$\ln(2) = \ln\left(\frac{6}{3}\right) = \ln(6) - \ln(3) \approx 1.8 - 1.1 = 0.7$

$\boxed{\ln(2) \approx 0.7}$

(b) $\ln(27) =$

$\ln(27) = \ln\left(3^3\right) = 3\ln(3) \approx 3(1.1) = 3.3$

$\boxed{\ln(27) \approx 3.3}$

(c) $\ln(18) =$

$\ln(18) = \ln(3 \cdot 6) = \ln(3) + \ln(6) \approx (1.1) + 1.8 = 2.9$

$\boxed{\ln(18) \approx 2.9}$

(d) $\ln\left(\sqrt{6}\right) =$

$\ln\left(\sqrt{6}\right) = \ln\left(6^{\frac{1}{2}}\right) = \frac{1}{2}\ln(6) \approx \frac{1}{2}(1.1) = 0.55$

$\boxed{\ln\left(\sqrt{6}\right) \approx 0.55}$

26. Compute: $\int \frac{1}{\sqrt{9-2x^2}} dx =$

This fits the form: $\int \frac{1}{\sqrt{a^2-u^2}} du = \arcsin\left(\frac{u}{a}\right) + C$

Here,

$$\begin{array}{lll} & a^2 & = 9 \\ \Rightarrow & a & = 3 \\ & u^2 & = 2x^2 \\ \Rightarrow & u & = \sqrt{2}x \\ \Rightarrow & \frac{du}{dx} & = \sqrt{2} \\ \Rightarrow & du & = \sqrt{2}dx \\ \Rightarrow & \frac{1}{\sqrt{2}}du = dx & \end{array}$$

Therefore, $\int \frac{1}{\sqrt{9-2x^2}} dx = \int \frac{1}{\sqrt{a^2-u^2}} \frac{1}{\sqrt{2}} du = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{a^2-u^2}} du = \frac{1}{\sqrt{2}} \arcsin\left(\frac{u}{a}\right) + C = \frac{1}{\sqrt{2}} \arcsin\left(\frac{\sqrt{2}x}{3}\right) + C$

$$\boxed{\int \frac{1}{\sqrt{9-2x^2}} dx = \frac{1}{\sqrt{2}} \arcsin\left(\frac{\sqrt{2}x}{3}\right) + C}$$

27. Compute:

$$\int \frac{1}{\sec(x)\sqrt{\sin(x)}}\,dx \quad \underset{\nwarrow \text{ re-write } \nearrow}{=} \quad \int \frac{1}{(\sin(x))^{\frac{1}{2}}}\frac{1}{\sec(x)}\,dx \quad \underset{\nwarrow \text{ re-write } \nearrow}{=} \quad \int (\sin(x))^{-\frac{1}{2}} \cos(x)\ dx$$

(a) 1. Is u-sub appropriate?

a. Is there a composite function?

Yes. $(\sin(x))^{-\frac{1}{2}}$

Let $u = \sin(x)$

b. Is there an "approximate function/derivative pair"?

Yes. $\sin(x) \to \cos(x)$

Let $u = \sin(x)$

2. Compute du

$$\boxed{\begin{array}{l} u = \sin(x) \\ \frac{du}{dx} = \cos(x) \\ du = \cos(x)\,dx \end{array}}$$

3. Analyze in terms of u and du

$$\int \underbrace{(\sin(x))^{-\frac{1}{2}}}_{u^{-\frac{1}{2}}}\underbrace{\cos(x)\ dx}_{du} = \int u^{-\frac{1}{2}}\,du$$

4. Integrate in terms of u

$$\int u^{-\frac{1}{2}}\,du = \frac{u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + C = 2u^{\frac{1}{2}} + C$$

5. Re-write in terms of x

$$\int \frac{1}{\sec(x)\sqrt{\sin(x)}}\,dx \ = \underbrace{2(\sin(x))^{\frac{1}{2}} + C}_{2u^{\frac{1}{2}}+C}$$

$$\boxed{\int \frac{1}{\sec(x)\sqrt{\sin(x)}}\,dx \ = 2(\sin(x))^{\frac{1}{2}} + C}$$

28. Compute: $\int \frac{\sin(x)\cos(x)}{\sin^2(x)-\cos^2(x)}dx \quad = \quad \int \frac{1}{\sin^2(x)-\cos^2(x)}\left(\sin(x)\cos(x)\right)\,dx$

$\nwarrow$ re-write $\nearrow$

(a) 1. Is u-sub appropriate?

a. Is there a composite function?

Yes. $\frac{1}{\sin^2(x)-\cos^2(x)} = \left(\sin^2(x)-\cos^2(x)\right)^{-1}$

Let $u = \sin^2(x) - \cos^2(x)$

b. Is there an "approximate function/derivative pair"? (This is a tricky one!)

Yes. $\left(\sin^2(x)-\cos^2(x)\right) \rightarrow \left(\sin(x)\cos(x)\right)$

Let $u = \sin^2(x) - \cos^2(x)$

2. Compute du

$$\begin{aligned} & u = \sin^2(x) - \cos^2(x) \\ & \frac{du}{dx} = 2\sin(x)\cos(x) - 2\cos(x)\left(-\sin(x)\right) = 4\sin(x)\cos(x) \\ & du = 4\sin(x)\cos(x)\ dx \\ & \frac{1}{4}du = \sin(x)\cos(x)\ dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\frac{1}{\sin^2(x)-\cos^2(x)}}_{\frac{1}{u}}\underbrace{\left(\sin(x)\cos(x)\right)dx}_{\frac{1}{4}du} = \int \frac{1}{u}\left(\frac{1}{4}du\right) = \frac{1}{4}\int \frac{1}{u}du$$

4. Integrate in terms of u

$$\frac{1}{4}\int \frac{1}{u}du = \frac{1}{4}\left[\ln|u|\right] + C = \frac{1}{4}\ln|u| + C$$

5. Re-write in terms of x

$$\int \frac{\sin(x)\cos(x)}{\sin^2(x)-\cos^2(x)}dx = \underbrace{\frac{1}{4}\ln\left|\sin^2(x)-\cos^2(x)\right| + C}_{\frac{1}{4}\ln|u|+C}$$

$$\boxed{\int \frac{\sin(x)\cos(x)}{\sin^2(x)-\cos^2(x)}dx = \frac{1}{4}\ln\left|\sin^2(x)-\cos^2(x)\right| + C}$$

29. Compute: $\underbrace{\frac{d}{dx}\left[\operatorname{arccsc}\left(x^2\right)\right]}_{\frac{d}{dx}[\operatorname{arccsc}(u)]} = \underbrace{-\frac{1}{|x^2|\sqrt{(x^2)^2-1}}}_{-\frac{1}{|u|\sqrt{u^2-1}}} \cdot \underbrace{2x}_{\frac{du}{dx}} = -\frac{2x}{x^2\sqrt{x^4-1}} = -\frac{2}{x\sqrt{x^4-1}}$

$$\boxed{\frac{d}{dx}\left[\operatorname{arccsc}\left(x^2\right)\right] = -\frac{2}{x\sqrt{x^4-1}}}$$