MTH 1126 Practice Test #1 - Solutions

SPRING 2017

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1. Compute: $\int \left(5x^4 + 4x^3 + 6x + 6\right) dx =$

$\int \left(5x^4 + 4x^3 + 6x + 6\right) dx = 5\left[\frac{x^5}{5}\right] + 4\left[\frac{x^4}{4}\right] + 6\left[\frac{x^2}{2}\right] + 6x + C$

i.e., $\int \left(5x^4 + 4x^3 + 6x + 6\right) dx = x^5 + x^4 + 3x^2 + 6x + C$ (Don't forget the "+C")

2. Compute: $\int (\sin(x) + \sec(x)\tan(x))\, dx =$

$\int (\sin(x) + \sec(x)\tan(x))\, dx = [-\cos(x)] + [\sec(x)] + C$

i.e., $\int (\sin(x) + \sec(x)\tan(x))\, dx = -\cos(x) + \sec(x) + C$ (Don't forget the "+C")

3. Compute: $\int_{x=1}^{x=2} \left(6x^3 + 4x^2 + 4x\right) dx =$

$$\int_{x=1}^{x=2} \underbrace{\left(6x^3 + 4x^2 + 4x\right)}_{f(x)} dx = \underbrace{\left[\frac{3}{2}x^4 + \frac{4}{3}x^3 + 2x^2\right]_{x=1}^{x=2}}_{F(x)}$$

$$= \underbrace{\left[\frac{3}{2}(2)^4 + \frac{4}{3}(2)^3 + 2(2)^2\right]}_{F(2)} - \underbrace{\left[\frac{3}{2}(1)^4 + \frac{4}{3}(1)^3 + 2(1)^2\right]}_{F(1)} = \frac{227}{6}$$

i.e., $\int_{x=1}^{x=2} \left(6x^3 + 4x^2 + 4x\right) dx = \frac{227}{6}$

4. Compute: $\int \left(8x^3+12x^2\right)^{10}\left(x^2+x\right)dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(8x^3+12x^2\right)^{10}$ (A function raised to a power is always a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(8x^3+12x^2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(8x^3+12x^2\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(x^2+x\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(8x^3+12x^2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 8x^3+12x^2 \\ \Rightarrow \tfrac{du}{dx} &= 24x^2+24x \\ \Rightarrow du &= \left(24x^2+24x\right)dx \\ \Rightarrow \tfrac{1}{24}du &= \left(x^2+x\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(8x^3+12x^2\right)^{10}}_{u^{10}}\underbrace{\left(x^2+x\right)dx}_{\frac{1}{24}du} = \int u^{10}\tfrac{1}{24}du = \tfrac{1}{24}\int u^{10}du$$

4. Integrate (in terms of u).

$$\tfrac{1}{24}\int u^{10}du = \tfrac{1}{24}\left[\tfrac{u^{11}}{11}\right]+C = \tfrac{1}{264}u^{11}+C$$

5. Re-express in terms of the original variable, x.

$$\int \left(8x^3+12x^2\right)^{10}\left(x^2+x\right)dx = \underbrace{\frac{1}{264}\left(8x^3+12x^2\right)^{11}+C}_{\frac{1}{264}u^{11}+C}$$

i.e., $\int \left(8x^3+12x^2\right)^{10}\left(x^2+x\right)dx = \frac{1}{264}\left(8x^3+12x^2\right)^{11}+C$

5. Compute: $\int \sin\left(x^3+3x^2\right)\left(6x^2+12x\right)dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\sin\left(x^3+3x^2\right)$

   $\nearrow$ outer $\qquad \nwarrow$ inner

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(x^3+3x^2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(x^3+3x^2\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(6x^2+12x\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(x^3+3x^2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= x^3+3x^2 \\ \Rightarrow \frac{du}{dx} &= 3x^2+6x \\ \Rightarrow du &= \left(3x^2+6x\right)dx \\ \Rightarrow 2du &= \left(6x^2+12x\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\sin\left(x^3+3x^2\right)}_{\sin(u)}\underbrace{\left(6x^2+12x\right)dx}_{2du} = \int \sin(u)\ 2du = 2\int \sin(u)\,du$$

4. Integrate (in terms of u).

$$2\int \sin(u)\,du = 2\left[-\cos(u)\right] + C = -2\cos(u) + C$$

5. Re-express in terms of the original variable, x.

$$\int \sin\left(x^3+3x^2\right)\left(6x^2+12x\right)dx = \underbrace{-2\cos\left(x^3+3x^2\right)+C}_{-2\cos(u)+C}$$

i.e., $\int \sin\left(x^3+3x^2\right)\left(6x^2+12x\right)dx = -2\cos\left(x^3+3x^2\right)+C$

6. Compute: $\int \frac{x+1}{3x^2+6x}dx =$

$$\int \frac{x+1}{3x^2+6x}dx \underbrace{=}_{\text{re-write}} \int \frac{1}{3x^2+6x}(x+1)\,dx$$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{3x^2+6x}$ is the same as $\left(3x^2+6x\right)^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(3x^2+6x\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(3x^2+6x\right)}_{\text{function}} - - - - \rightarrow \underbrace{(x+1)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(3x^2+6x\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 3x^2+6x \\ \Rightarrow \frac{du}{dx} &= 6x+6 \\ \Rightarrow du &= (6x+6)\,dx \\ \Rightarrow \frac{1}{6}du &= (x+1)\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\frac{1}{3x^2+6x}}_{\frac{1}{u}} \underbrace{(x+1)\,dx}_{\frac{1}{6}du} = \int \frac{1}{u}\cdot\frac{1}{6}du = \frac{1}{6}\int \frac{1}{u}du$$

4. Integrate (in terms of u).

$$\frac{1}{6}\int \frac{1}{u}du = \frac{1}{6}\left[\ln|u|\right] + C = \frac{1}{6}\ln|u| + C$$

5. Re-express in terms of the original variable, x.

$$\int \frac{x+1}{3x^2+6x}dx = \underbrace{\frac{1}{6}\ln\left|3x^2+6x\right| + C}_{\frac{1}{6}\ln|u|+C}$$

i.e., $\int \frac{x+1}{3x^2+6x}dx = \frac{1}{6}\ln\left|3x^2+6x\right| + C$

7. Compute: $\frac{d}{dx}\left[\ln\left(\sin\left(x\right)\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(\sin\left(x\right)\right)\right]}_{\frac{d}{dx}\left[\ln(g(x))\right]} = \underbrace{\frac{1}{\sin\left(x\right)}}_{\frac{1}{g(x)}} \cdot \underbrace{\cos\left(x\right)}_{g'(x)} = \frac{\cos(x)}{\sin(x)} = \cot\left(x\right)$$

i.e., $\frac{d}{dx}\left[\ln\left(\sin\left(x\right)\right)\right] = \frac{\cos(x)}{\sin(x)} = \cot\left(x\right)$

8. Compute: $\frac{d}{dx}\left[\ln\left(3x^3 - 9x + 5\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(3x^3 - 9x + 5\right)\right]}_{\frac{d}{dx}\left[\ln(g(x))\right]} = \underbrace{\frac{1}{3x^3 - 9x + 5}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(9x^2 - 9\right)}_{g'(x)} = \frac{9x^2-9}{3x^3-9x+5}$$

i.e., $\frac{d}{dx}\left[\ln\left(3x^3 - 9x + 5\right)\right] = \frac{9x^2-9}{3x^3-9x+5}$

9. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2-1}{x}}\right)\right] \underbrace{=}_{\text{re-write}} \frac{d}{dx}\left[\ln\left[\left(\frac{x^2-1}{x}\right)^{\frac{1}{2}}\right]\right]$

Remark: We can compute this derivative directly, in it's current form, but it would be much easier to use the *algebraic properties* of natural logarithms to simplify the expression first.

$$\frac{d}{dx}\left[\ln\left[\left(\frac{x^2-1}{x}\right)^{\frac{1}{2}}\right]\right] = \underbrace{\frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{x^2-1}{x}\right)\right]}_{\ln(a^n) = n\ln(a)} = \underbrace{\frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(x^2-1\right) - \ln\left(x\right)\right)\right]}_{\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)} = \frac{1}{2}\frac{d}{dx}\left[\ln\left(x^2-1\right) - \ln\left(x\right)\right]$$

NOW we're ready to compute the derivative!

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2-1}{x}}\right)\right] = \frac{1}{2}\frac{d}{dx}\left[\ln\left(x^2-1\right) - \ln\left(x\right)\right] = \frac{1}{2}\left[\frac{1}{x^2-1}\left(2x\right) - \frac{1}{x}\right] = \frac{x}{x^2-1} - \frac{1}{2x}$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2-1}{x}}\right)\right] = \frac{x}{x^2-1} - \frac{1}{2x}$

10. Compute: $\int_{x=-1}^{x=1}\left(x^2-3x+1\right)^3(8x-12)\,dx=$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(x^2-3x+1\right)^3$ (A function raised to a power is *always* a composite function!)

   Let $u=$ the "inner" of the composite function

   $\Rightarrow u=\left(x^2-3x+1\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(x^2-3x+1\right)}_{\text{function}} - - - - \rightarrow \underbrace{(8x-12)}_{\text{deriv}}$

   Let $u=$ the "function" of the function/deriv pair

   $\Rightarrow u=\left(x^2-3x+1\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= x^2-3x+1 \\ \Rightarrow \frac{du}{dx} &= 2x-3 \\ \Rightarrow du &= (2x-3)\,dx \\ \Rightarrow 4du &= (8x-12)\,dx \end{aligned}$$

When $x=-1$, $u=x^2-3x+1=(-1)^2-3(-1)+1=5$
When $x=1$, $u=x^2-3x+1=(1)^2-3(1)+1=-1$

3. Analyze in terms of u and du

$$\int_{x=-1}^{x=1}\underbrace{\left(x^2-3x+1\right)^3}_{u^3}\underbrace{(8x-12)\,dx}_{4du}=\int_{u=5}^{u=-1}u^3\cdot 4du=4\int_{u=5}^{u=-1}u^3du$$

   Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$4\int_{u=5}^{u=-1}u^3du=4\left[\frac{u^4}{4}\right]_{u=5}^{u=-1}=\left[u^4\right]_{u=5}^{u=-1}=\underbrace{(-1)^4}_{F(-1)}-\underbrace{(5)^4}_{F(5)}=-624$$

i.e., $\int_{x=-1}^{x=1}\left(x^2-3x+1\right)^3(8x-12)\,dx=-624$

11. Compute: $\int \frac{\cos x+2x^2}{3\sin(x)+2x^3}dx =$

$$\int \frac{\cos x+2x^2}{3\sin(x)+2x^3}dx \underbrace{=}_{\text{re-write}} \int \frac{1}{3\sin(x)+2x^3}\left(\cos x+2x^2\right)dx$$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{3\sin(x)+2x^3}$ is the same as $\left(3\sin(x)+2x^3\right)^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(3\sin(x)+2x^3\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(3\sin(x)+2x^3\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(\cos x+2x^2\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(3\sin(x)+2x^3\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 3\sin(x)+2x^3 \\ \Rightarrow \frac{du}{dx} &= 3\cos(x)+6x^2 \\ \Rightarrow du &= \left(3\cos(x)+6x^2\right)dx \\ \Rightarrow \frac{1}{3}du &= \left(\cos(x)+2x^2\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\frac{1}{3\sin(x)+2x^3}}_{\frac{1}{u}}\underbrace{\left(\cos x+2x^2\right)dx}_{\frac{1}{3}du} = \int \frac{1}{u}\cdot\frac{1}{3}du = \frac{1}{3}\int\frac{1}{u}du$$

4. Integrate (in terms of u).

$$\frac{1}{3}\int\frac{1}{u}du = \frac{1}{3}\left[\ln|u|\right]+C = \frac{1}{3}\ln|u|+C$$

5. Re-express in terms of the original variable, x.

$$\int \frac{\cos x+2x^2}{3\sin(x)+2x^3}dx = \underbrace{\frac{1}{3}\ln\left|3\sin(x)+2x^3\right|+C}_{\frac{1}{3}\ln|u|+C}$$

i.e., $\int \frac{\cos x+2x^2}{3\sin(x)+2x^3}dx = \frac{1}{3}\ln\left|3\sin(x)+2x^3\right|+C$

12. Compute: $\frac{d}{dx}\left[e^{\cos(x)}\right] =$

$$\underbrace{\frac{d}{dx}\left[e^{\cos(x)}\right]}_{\frac{d}{dx}[e^u]} = \underbrace{e^{\cos(x)}}_{e^u} \cdot \underbrace{(-\sin(x))}_{\frac{du}{dx}} = -\sin(x)\, e^{\cos(x)}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[e^{\cos(x)}\right] = -\sin(x)\, e^{\cos(x)}}$$

13. Compute: $\int \frac{e^x}{\sqrt{4-e^{2x}}} dx \quad = \quad \int \frac{1}{\sqrt{4-e^{2x}}} \cdot e^x\, dx$

↖ re-write ↗

$\int \frac{1}{\sqrt{4-e^{2x}}} \cdot e^x\, dx$ appears to fit the form: $\int \frac{1}{\sqrt{a^2-u^2}} du$

If this analysis is correct, then:

$$\boxed{\begin{array}{lcl} a^2 & = & 4 \\ a & = & 2 \\ u^2 & = & e^{2x} = (e^x)^2 \\ u & = & e^x \\ \frac{du}{dx} & = & e^x \\ du & = & e^x dx \end{array}}$$

Now analyze the integral in terms of u and du.

$$\int \frac{1}{\sqrt{4-e^{2x}}} \cdot e^x\, dx = \int \underbrace{\frac{1}{\sqrt{(2)^2-(e^x)^2}}}_{\frac{1}{\sqrt{a^2-u^2}}} \underbrace{e^x dx}_{du} = \int \frac{1}{\sqrt{a^2-u^2}} du$$

Integrate:

$\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1}\left(\frac{u}{a}\right) = \sin^{-1}\left(\frac{e^x}{2}\right) + C$

$$\boxed{\text{i.e., } \int \frac{e^x}{\sqrt{4-e^{2x}}} dx = \sin^{-1}\left(\frac{e^x}{2}\right) + C}$$

14. Given that $\ln(2) \approx 0.7$ and $\ln(5) \approx 1.6$, approximate the following:

   (a) $\ln(10) =$

   $\ln(10) = \ln(2\cdot 5) = \ln(2) + \ln(5) \approx 0.7 + 1.6 = 2.3$

   $\boxed{\ln(10) \approx 2.3}$

   (b) $\ln(50) =$

   $\ln(50) = \ln(2\cdot 5^2) = \ln(2) + \ln(5^2) = \ln(2) + 2\ln(5) \approx 0.7 + 2(1.6) = 3.9$

   $\boxed{\ln(50) \approx 3.9}$

15. $\int e^{3x^2} x\,dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes. e^{3x^2}

   Let $u = 3x^2$ i.e., "Let $u =$ the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

   Yes. $\underbrace{3x^2}_{\text{function}} \rightarrow \underbrace{x}_{\text{deriv}}$

   Let $u = 3x^2$ i.e., "Let $u =$ the 'inner' function"

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & 3x^2 \\ \Rightarrow & \frac{du}{dx} & = & 6x \\ \Rightarrow & du & = & 6x\,dx \\ \Rightarrow & \frac{1}{6}du & = & x\,dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{e^{3x^2}}_{e^u} \underbrace{x\,dx}_{\frac{1}{6}du} = \int e^u \tfrac{1}{6} du = \tfrac{1}{6} \int e^u\,du$$

4. Integrate in terms of u

$$\tfrac{1}{6} \int e^u\,du = \tfrac{1}{6} e^u + C$$

5. Re-write in terms of x

$$\int e^{3x^2} x\,dx = \underbrace{\frac{1}{6} e^{3x^2} + C}_{\frac{1}{6}e^u + C}$$

$$\boxed{\text{i.e., } \int e^{3x^2} x\,dx = \tfrac{1}{6} e^{3x^2} + C}$$

16. $\frac{d}{dx}\left[\tan^{-1}(\sin(x))\right] =$

$$\underbrace{\frac{d}{dx}\left[\tan^{-1}(\sin(x))\right]}_{\frac{d}{dx}[\tan^{-1}(u)]} = \underbrace{\frac{1}{1 + (\sin(x))^2}}_{\frac{1}{1+u^2}} \cdot \underbrace{\cos(x)}_{\frac{du}{dx}} = \frac{\cos(x)}{1 + \sin^2(x)}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[\tan^{-1}(\sin(x))\right] = \frac{\cos(x)}{1 + \sin^2(x)}}$$

17. Write the given equation in algebraic form.

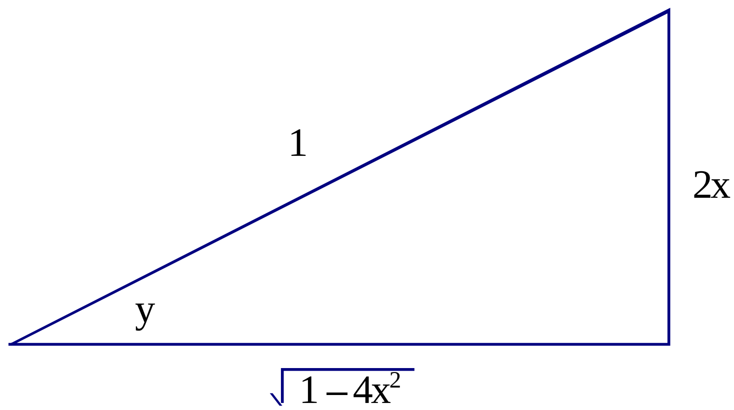
$z = \cos(\arcsin(2x))$

Let $y = \arcsin(2x)$

This is the same as saying "y is the angle whose sine is $2x$."

i.e., $\sin(y) = 2x$.

Let's draw a triangle to depict this relationship.



By Pythagorean's Theorem, $(opp)^2 + (adj)^2 = (hyp)^2$
$\Rightarrow \sqrt{(hyp)^2 - (opp)^2} = adj \Rightarrow adj = \sqrt{1-4x^2}$.

Recall: we want $z = \cos(\arcsin(2x)) = \cos(y)$

From the picture, $z = \cos(y) = \frac{adj}{hyp} = \frac{\sqrt{1-4x^2}}{1} = \sqrt{1-4x^2}$

Hence, $z = \cos(\arcsin(2x)) = \sqrt{1-4x^2}$

$\boxed{\text{i.e., } z = \sqrt{1-4x^2}}$

18. $\int \frac{4x^2}{(4x^3+6)^{\frac{3}{2}}} dx \quad = \quad \int \frac{1}{(4x^3+6)^{\frac{3}{2}}} \cdot 4x^2\, dx \quad = \quad \int \left(4x^3+6\right)^{-\frac{3}{2}} \cdot 4x^2\, dx$

$\nwarrow$ re-write $\nearrow$ $\qquad$ $\nwarrow$ re-write $\nearrow$

(a) 1. Is u-sub appropriate?

a. Is there a composite function?

Yes. $\left(4x^3+6\right)^{-\frac{3}{2}}$

Let $u = 4x^3 + 6$

b. Is there an "approximate function/derivative pair"?

Yes. $\left(4x^3+6\right) \to 4x^2$

Let $u = x^2 + 1$

2. Compute du

$$\boxed{\begin{array}{l} u = 4x^3 + 6 \\ du = 12x^2 dx \\ \frac{1}{3} du = 4x^2 dx \end{array}}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(4x^3+6\right)^{-\frac{3}{2}}}_{u^{-\frac{3}{2}}} \underbrace{4x^2\, dx}_{\frac{1}{3} du} = \int u^{-\frac{3}{2}} \left(\tfrac{1}{3} du\right) = \tfrac{1}{3} \int u^{-\frac{3}{2}} du$$

4. Integrate in terms of u

$$\tfrac{1}{3} \int u^{-\frac{3}{2}} du = \tfrac{1}{3} \left[\frac{u^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)} \right] + C = \tfrac{1}{3} \left(-2\right) u^{-\frac{1}{2}} + C = -\tfrac{2}{3} u^{-\frac{1}{2}} + C$$

5. Re-write in terms of x

$$\int \frac{4x^2}{(4x^3+6)^{\frac{3}{2}}} dx = \underbrace{-\frac{2}{3} \left(4x^3+6\right)^{-\frac{1}{2}} + C}_{-\frac{2}{3} u^{-\frac{1}{2}} + C}$$

$$\boxed{\int \frac{4x^2}{(4x^3+6)^{\frac{3}{2}}} dx = -\tfrac{2}{3} \left(4x^3+6\right)^{-\frac{1}{2}} + C}$$

19. Write the given equation in algebraic form.

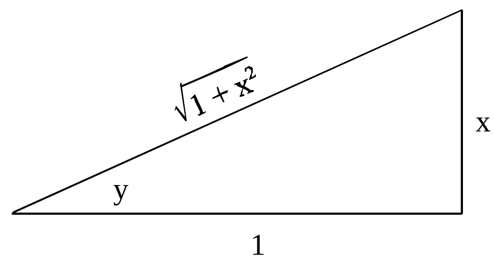
$z = \sec\left(\tan^{-1}(x)\right)$

Let $y = \tan^{-1}(x)$

This is the same as saying "y is the angle whose tangent is x."

i.e., $\tan(y) = x = \frac{\text{opp}}{\text{adj}} = \frac{x}{1}$.

Let's draw a triangle to depict this relationship.



By Pythagorean's Theorem, $(opp)^2 + (adj)^2 = (hyp)^2 \Rightarrow 1^2 + x^2 = (hyp)^2$

$\Rightarrow (hyp)^2 = \sqrt{1+x^2}$

Recall: we want $z = \sec\left(\tan^{-1}(x)\right) = \sec(y)$.

From the picture, $z = \sec(y) = \frac{hyp}{adj} = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}$.

$$\boxed{z = \sqrt{1+x^2}}$$

20. $\underbrace{\frac{d}{dx}\left[\arccos(3x-\pi)\right]}_{\frac{d}{dx}[\arccos(u)]} = \underbrace{-\frac{1}{\sqrt{1-(3x-\pi)^2}}}_{-\frac{1}{\sqrt{1-u^2}}} \cdot \underbrace{3}_{\frac{du}{dx}} = -\frac{3}{\sqrt{1-(3x-\pi)^2}}$

$$\boxed{\frac{d}{dx}\left[\arccos(3x-\pi)\right] = -\frac{3}{\sqrt{1-(3x-\pi)^2}}}$$

21. $\int_0^1 \frac{1}{\sqrt{4-x^2}} dx =$

This fits the form: $\int \frac{1}{\sqrt{a^2-u^2}} du = \arcsin\left(\frac{u}{a}\right) + C$

Here,

$$\begin{array}{|lcl|} \hline & a^2 & = 4 \\ \Rightarrow & a & = 2 \\ & u^2 & = x^2 \\ \Rightarrow & u & = x \\ \Rightarrow & \frac{du}{dx} & = 1 \\ \Rightarrow & du & = dx \\ \multicolumn{3}{|l|}{\text{when } x = 0;\ u = x = 0} \\ \multicolumn{3}{|l|}{\text{when } x = 1;\ u = x = 1} \\ \hline \end{array}$$

Therefore, $\int_{x=0}^{x=1} \frac{1}{\sqrt{4-x^2}} dx = \int_{u=0}^{u=1} \frac{1}{\sqrt{a^2-u^2}} dx = \left[\arcsin\left(\frac{u}{a}\right)\right]_{u=0}^{u=1} = \left[\arcsin\left(\frac{u}{2}\right)\right]_{u=0}^{u=1}$

$$= \arcsin\left(\frac{1}{2}\right) = \arcsin(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

$$\boxed{\int_{x=0}^{x=1} \frac{1}{\sqrt{4-x^2}} dx = \frac{\pi}{6}}$$

22. $\int \frac{\sec^2(x)}{\sqrt{\tan^3(x)}}dx \underset{\nwarrow \text{ re-write } \nearrow}{=} \int \frac{1}{(\tan^3(x))^{\frac{1}{2}}} \cdot \sec^2(x)\ dx \underset{\nwarrow \text{ re-write } \nearrow}{=} \int (\tan(x))^{-\frac{3}{2}} \sec^2(x)\, dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $(\tan(x))^{-\frac{3}{2}}$ (A function raised to a power is always a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \tan(x)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\tan(x)}_{\text{function}} - - - - \rightarrow \underbrace{\sec^2(x)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \tan(x)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= \tan(x) \\ \Rightarrow \frac{du}{dx} &= \sec^2(x) \\ \Rightarrow du &= \sec^2(x)\, dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{(\tan(x))^{-\frac{3}{2}}}_{u^{-\frac{3}{2}}} \underbrace{\sec^2(x)\ dx}_{du} = \int u^{-\frac{3}{2}} du$$

4. Integrate (in terms of u).

$$\int u^{-\frac{3}{2}} du = \frac{u^{-\frac{1}{2}}}{-\frac{1}{2}} + C = -2u^{-\frac{1}{2}} + C$$

5. Re-express in terms of the original variable, x.

$$\boxed{\int \frac{\sec^2(x)}{\sqrt{\tan^3(x)}}dx = -2(\tan(x))^{-\frac{1}{2}} + C = -\frac{2}{\sqrt{\tan(x)}} + C}$$

23. $\frac{d}{dx}\underbrace{\left[e^{\left(\tan\left(3x^2\right)\right)}\right]}_{e^u} = \underbrace{e^{\left(\tan\left(3x^2\right)\right)}}_{e^u} \cdot \underbrace{\sec^2\left(3x^2\right) \cdot 6x}_{\frac{du}{dx}}$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[e^{\left(\tan\left(3x^2\right)\right)}\right] = 6x\sec^2\left(3x^2\right) e^{\left(\tan\left(3x^2\right)\right)}}$$

24. $\int e^{\left(2x^2+7\right)} x dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $e^{\left(2x^2+7\right)}$ (A function raised to a power is always a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(2x^2+7\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(2x^2+7\right)}_{\text{function}} - - - - - \rightarrow \underbrace{x}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(2x^2+7\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 2x^2+7 \\ \Rightarrow \frac{du}{dx} &= 4x \\ \Rightarrow du &= 4xdx \\ \Rightarrow \frac{1}{4}du &= xdx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{e^{\left(2x^2+7\right)}}_{e^u} \underbrace{x\,dx}_{\frac{1}{4}du} = \int e^u \tfrac{1}{4} du = \tfrac{1}{4} \int e^u du$$

4. Integrate (in terms of u).

$$\tfrac{1}{4} \int e^u du = \tfrac{1}{4} e^u + C$$

5. Re-express in terms of the original variable, x.

$$\int e^{\left(2x^2+7\right)} x dx = \underbrace{\frac{1}{4} e^{\left(2x^2+7\right)} + C}_{\frac{1}{4}e^u + C}$$

i.e., $\int e^{\left(2x^2+7\right)} x dx = \frac{1}{4} e^{\left(2x^2+7\right)} + C$

25. $\frac{d}{dx}\left[\sin^{-1}\left(\sqrt{x}\right)\right] = \underbrace{\frac{d}{dx}\left[\sin^{-1}\left(x^{\frac{1}{2}}\right)\right]}_{\frac{d}{dx}[\arcsin(u)]} = \underbrace{\frac{1}{\sqrt{1-\left(x^{\frac{1}{2}}\right)^2}}}_{\frac{1}{\sqrt{1-u^2}}} \cdot \underbrace{\frac{1}{2}x^{-\frac{1}{2}}}_{\frac{du}{dx}} = \frac{1}{2x^{\frac{1}{2}}\sqrt{1-x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$

i.e., $\frac{d}{dx}\left[\sin^{-1}\left(\sqrt{x}\right)\right] = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$

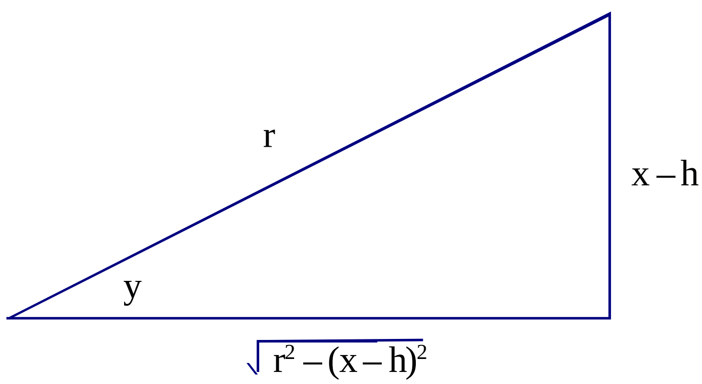
26. Write the given equation in algebraic form. $z = \cos\left(\arcsin\left(\frac{x-h}{r}\right)\right)$

Let $y = \arcsin\left(\frac{x-h}{r}\right)$

This is the same as saying "y is the angle whose sine is $\frac{x-h}{r}$."

i.e., $\sin(y) = \frac{x-h}{r}$.

Let's draw a triangle to depict this relationship.



By Pythagorean's Theorem, $(opp)^2 + (adj)^2 = (hyp)^2$

$\Rightarrow \sqrt{(hyp)^2 - (opp)^2} = adj \Rightarrow adj = \sqrt{r^2 - (x-h)^2}$.

Anyway, we want $z = \cos\left(\arcsin\left(\frac{x-h}{r}\right)\right) = \cos(y) = \frac{adj}{hyp} = \frac{\sqrt{r^2-(x-h)^2}}{r}$

i.e., $z = \frac{\sqrt{r^2-(x-h)^2}}{r}$

27. $\underbrace{\frac{d}{dx}\left[\arctan\left(3x^2\right)\right]}_{\frac{d}{dx}[\arctan(u)]} = \underbrace{\frac{1}{1+\left(3x^2\right)^2}}_{\frac{1}{1+u^2}} \cdot \underbrace{6x}_{\frac{du}{dx}} = \frac{6x}{1+9x^4}$

i.e., $\frac{d}{dx}\left[\arctan\left(3x^2\right)\right] = \frac{6x}{1+9x^4}$

28. $\int \frac{e^{2x}}{4+e^{4x}}dx \quad = \quad \int \frac{1}{(2)^2+(e^{2x})^2}e^{2x}dx$

↖ re-write ↗

$\int \frac{1}{(2)^2+(e^{2x})^2}e^{2x}dx$ appears to fit the form: $\int \frac{1}{a^2+u^2}du$

If this analysis is correct, then:

$a^2 = 4$
$\Rightarrow a = 2$
$u^2 = e^{4x}$
$\Rightarrow u = e^{2x}$
$\Rightarrow du = 2e^{2x}dx$
$\Rightarrow \frac{1}{2}du = e^{2x}dx$

Therefore, $\int \frac{e^{2x}}{4+e^{4x}}dx = \underbrace{\int \frac{1}{(2)^2+(e^{2x})^2}}_{\int \frac{1}{a^2+u^2}} \underbrace{e^{2x}dx}_{\frac{1}{2}du} = \int \frac{1}{a^2+u^2} \cdot \frac{1}{2}du = \frac{1}{2}\int \frac{1}{a^2+u^2}du = \frac{1}{2} \cdot \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$

$$= \frac{1}{2} \cdot \frac{1}{2}\arctan\left(\frac{e^{2x}}{2}\right) + C = \frac{1}{4}\arctan\left(\frac{e^{2x}}{2}\right) + C =$$

i.e., $\int \frac{e^{2x}}{4+e^{4x}}dx = \frac{1}{4}\arctan\left(\frac{e^{2x}}{2}\right) + C$

29. $\int_{\frac{2}{\sqrt{3}}}^{2} \frac{1}{x\sqrt{x^2-1}}dx =$

This fits the form: $\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{a}\,\mathrm{arcsec}\left(\frac{|u|}{a}\right) + C$

$$\begin{array}{lcl} & a^2 &=& 1 \\ \Rightarrow & a &=& 1 \\ & u^2 &=& x^2 \\ \Rightarrow & u &=& x \\ \Rightarrow & du &=& dx \end{array}$$
when $x = \frac{2}{\sqrt{3}}$; $u = x = \frac{2}{\sqrt{3}}$
when $x = 2$; $u = x = 2$

Therefore: $\int_{x=\frac{2}{\sqrt{3}}}^{x=2} \frac{1}{x\sqrt{x^2-1}}dx = \int_{u=\frac{2}{\sqrt{3}}}^{u=2} \frac{1}{u\sqrt{u^2-a^2}}du = \left[\frac{1}{a}\,\mathrm{arcsec}\left(\frac{|u|}{a}\right)\right]_{u=\frac{2}{\sqrt{3}}}^{u=2} = \left[\frac{1}{1}\,\mathrm{arcsec}\left(\frac{|u|}{1}\right)\right]_{u=\frac{2}{\sqrt{3}}}^{u=2}$

$$= \left[\mathrm{arcsec}\left(|u|\right)\right]_{u=\frac{2}{\sqrt{3}}}^{u=2} = \mathrm{arcsec}\left(2\right) - \mathrm{arcsec}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

i.e., $\int_{x=\frac{2}{\sqrt{3}}}^{x=2} \frac{1}{x\sqrt{x^2-1}}dx = \frac{\pi}{6}$

Remark 1 *How did I compute the values of* $\mathrm{arcsec}\,(2)$ *and* $\mathrm{arcsec}\left(\frac{2}{\sqrt{3}}\right)$?

Observe: $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \Rightarrow \sec\left(\frac{\pi}{3}\right) = 2 \Rightarrow \mathrm{arcsec}\,(2) = \frac{\pi}{3}$

Also: $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \Rightarrow \sec\left(\frac{\pi}{6}\right) = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}} \Rightarrow \mathrm{arcsec}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$

30. Compute: $\int \frac{e^{6x}+x}{e^{6x}+3x^2}\,dx =$

$$\int \frac{e^{6x}+x}{e^{6x}+3x^2}\,dx \underbrace{=}_{\text{re-write}} \int \frac{1}{e^{6x}+3x^2}\left(e^{6x}+x\right)dx$$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

      Yes! $\frac{1}{e^{6x}+3x^2}$ is the same as $\left(e^{6x}+3x^2\right)^{-1}$, so it is a function raised to a power.

      Let $u =$ the "inner" of the composite function

      $\Rightarrow u = \left(e^{6x}+3x^2\right)$

   b. Is there an (approximate) function/derivative pair?

      Yes! $\underbrace{\left(e^{6x}+3x^2\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(e^{6x}+x\right)}_{\text{deriv}}$

      Let $u =$ the "function" of the function/deriv pair

      $\Rightarrow u = \left(e^{6x}+3x^2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

      (i.e., do criteria **a** and **b** suggest the same choice of u ?)

      Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= e^{6x}+3x^2 \\ \Rightarrow \frac{du}{dx} &= 6e^{6x}+6x \\ \Rightarrow du &= \left(6e^{6x}+6x\right)dx \\ \Rightarrow \frac{1}{6}du &= \left(e^{6x}+x\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\frac{1}{e^{6x}+3x^2}}_{\frac{1}{u}}\underbrace{\left(e^{6x}+x\right)dx}_{\frac{1}{6}du} = \int \frac{1}{u}\cdot\frac{1}{6}du = \frac{1}{6}\int \frac{1}{u}du$$

4. Integrate (in terms of u).

$$\frac{1}{6}\int \frac{1}{u}du = \frac{1}{6}\left[\ln|u|\right] + C = \frac{1}{6}\ln|u| + C$$

5. Re-express in terms of the original variable, x.

$$\int \frac{e^{6x}+x}{e^{6x}+3x^2}\,dx = \underbrace{\frac{1}{6}\ln\left|e^{6x}+3x^2\right| + C}_{\frac{1}{6}\ln|u|+C}$$

$$\text{i.e., } \int \frac{e^{6x}+x}{e^{6x}+3x^2}\,dx = \frac{1}{6}\ln\left|e^{6x}+3x^2\right| + C$$