MTH 3318 - Test #2 - Solutions

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Instructions. Show your work completely. Document your work well.

Remark 1 *For problems 1 - 3, prove one.*

1. $A \cap B = A \Rightarrow A \subseteq B$

   Proof. Let the hypothesis be given. (i.e., let $A \cap B = A$).

   We need to show that $A \subseteq B$.

   So let $x \in A$

   $\Rightarrow x \in A \cap B$ (because $A = A \cap B$ by hypothesis).

   $\Rightarrow x \in A$ and $x \in B$

   In particular, $x \in B$.

   We have just shown that $x \in A \Rightarrow x \in B$.

   Hence, $A \subseteq B$ ■

2. $A \subseteq B \Rightarrow (A \cup B) = B$

Proof. Let the hypothesis be given. (i.e., let $A \subseteq B$)

We must show that:

a. $(A \cup B) \subseteq B$

and

b. $B \subseteq (A \cup B)$

a. $(A \cup B) \subseteq B$

Let $x \in (A \cup B)$.

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in B$ (because $A \subseteq B$ by hypothesis.)

i.e., $x \in B$

We have shown that $x \in (A \cup B) \Rightarrow x \in B$.

Therefore, $(A \cup B) \subseteq B$

b. $\boxed{B \subseteq (A \cup B) \text{ (This is } \textit{always} \text{ true.)}}$ ■

3. $A \subseteq B \Rightarrow B^c \subseteq A^c$

   Proof. Let the hypothesis be given. (i.e., let $A \subseteq B$).

   We need to show that $B^c \subseteq A^c$.

   So let $x \in B^c$

   $\Rightarrow x \notin B$

   $\Rightarrow x \notin A$ (Otherwise, if x *were* an element of A, then our hypothesis would imply that $\Rightarrow x \in B$, contradicting the fact that $x \notin B$.)

   $\Rightarrow x \in A^c$.

   We have shown that $x \in B^c \Rightarrow x \in A^c$.

   Hence, $B^c \subseteq A^c$. ■

Remark 2 *For problems 4 - 6, prove one.*

4. $(A \cap B)^c = A^c \cup B^c$

   Proof. We must show that:

   (a) $(A \cap B)^c \subseteq A^c \cup B^c$

   and

   (b) $A^c \cup B^c \subseteq (A \cap B)^c$

   a. $\boxed{(A \cap B)^c \subseteq A^c \cup B^c}$

   Let $x \in (A \cap B)^c$

   $\Rightarrow x \notin (A \cap B)$

   $\Rightarrow x \notin A$ or $x \notin B$

   $\Rightarrow x \in A^c$ or $x \in B^c$

   $\Rightarrow x \in A^c \cup B^c$

   We have shown that $x \in (A \cap B)^c \Rightarrow x \in A^c \cup B^c$

   Therefore, $(A \cap B)^c \subseteq A^c \cup B^c$

b. $A^c \cup B^c \subseteq (A \cap B)^c$

Let $x \in A^c \cup B^c$

$\Rightarrow x \in A^c$ or $x \in B^c$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \in (A \cap B)^c$

We have shown that $x \in A^c \cup B^c \Rightarrow x \in (A \cap B)^c$

Therefore, $A^c \cup B^c \subseteq (A \cap B)^c$ ■

5. $A \subseteq B \Rightarrow (A \cap B) = A$

Let the hypothesis be given. (i.e., let $A \subseteq B$)

We must show that:

a. $(A \cap B) \subseteq A$ (This is *always* true.)

and

b. $A \subseteq (A \cap B)$

Let $x \in A$.

$\Rightarrow x \in B$ (Because $A \subseteq B$, by hypothesis).

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in A \cap B$

We have shown that $x \in A \Rightarrow x \in A \cap B$

Therefore, $A \subseteq (A \cap B)$ ■

6. $(A \cup B)^c = A^c \cap B^c$

Proof. We must show that:

(a) $(A \cup B)^c \subseteq A^c \cap B^c$

and

(b) $A^c \cap B^c \subseteq (A \cup B)^c$

a. $\boxed{(A \cup B)^c \subseteq A^c \cap B^c}$

Let $x \in (A \cup B)^c$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A^c$ and $x \in B^c$

$\Rightarrow x \in A^c \cap B^c$

i.e., $x \in (A \cup B)^c \Rightarrow x \in A^c \cap B^c$

Thus, $(A \cup B)^c \subseteq A^c \cap B^c$

b. $\boxed{A^c \cap B^c \subseteq (A \cup B)^c}$

Let $x \in A^c \cap B^c$

$\Rightarrow x \in A^c$ and $x \in B^c$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \in (A \cup B)^c$

i.e., $x \in A^c \cap B^c \Rightarrow x \in (A \cup B)^c$

Thus, $A^c \cap B^c \subseteq (A \cup B)^c$ ■

Remark 3 *Prove problem 7.*

7. $A \cap B = \emptyset \Leftrightarrow (B \cap A^c) = B$

   Proof. We must show:

   (a) $A \cap B = \emptyset \Rightarrow (B \cap A^c) = B$

   and

   (b) $(B \cap A^c) = B \Rightarrow A \cap B = \emptyset$

   a. $\boxed{A \cap B = \emptyset \Rightarrow (B \cap A^c) = B}$

   Let the hypothesis be given (i.e., suppose that $A \cap B = \emptyset$)

   We must show:

   i. $(B \cap A^c) \subseteq B$ (This is *always* true.)

   and

   ii. $B \subseteq (B \cap A^c)$

   Let $x \in B$

   $\Rightarrow x \notin A$ (because $A \cap B = \emptyset$ by hypothesis)

   $\Rightarrow x \in A^c$

   i.e., $x \in B$ and $x \in A^c$

   $\Rightarrow x \in B \cap A^c$

   i.e., $x \in B \Rightarrow x \in B \cap A^c$

   Hence, $B \subseteq (B \cap A^c)$

b. $\boxed{(B \cap A^c) = B \Rightarrow A \cap B = \emptyset}$

Let the hypothesis be given (i.e., suppose that $(B \cap A^c) = B$

To show that $A \cap B = \emptyset$, we must either show that $x \in B \Rightarrow x \notin A$ or that $x \in A \Rightarrow x \notin B$

We will choose the latter: $x \in B \Rightarrow x \notin A$

Let $x \in B$

$\Rightarrow x \in (B \cap A^c)$ (Because $(B \cap A^c) = B$, by our hypothesis)

$\Rightarrow x \in B$ and $x \in A^c$

Specifically, $x \in A^c$

$\Rightarrow x \notin A$

i.e., $x \in B \Rightarrow x \notin A$

Thus, $A \cap B = \emptyset$ ■

Remark 4 *For problems 8 - 9, prove either one by contradiction.*

8. $\underbrace{U^c = \emptyset}_{p}$

   Proof. (By contradiction). Suppose, for the sake of deriving a contradiction, that $\underbrace{U^c \neq \emptyset}_{\sim p}$.

   $\Rightarrow \exists x \in U^c$

   $\Rightarrow \underbrace{x \notin U}_{q}$

   This contradicts the definition of universe: $\underbrace{x \in U \ \forall x}_{\sim q}$.

   (Thus, we have the contradiction: $\underbrace{x \notin U}_{q} \wedge \underbrace{x \in U}_{\sim q}$)

   Since the assumption that $U^c \neq \emptyset$ leads to a contradiction, it must be false.

   Hence, $U^c = \emptyset$. ■

9. $(A \cap B) = \emptyset \Rightarrow \underbrace{A \subseteq B^c}_{p}$

Proof. (By contradiction) Let the hypothesis be given. (i.e., suppose that $(A \cap B) = \emptyset$.)

Suppose, for the sake of deriving a contradiction, that $\underbrace{A \nsubseteq B^c}_{\sim p}$.

$\Rightarrow \exists x \in A$ such that $x \notin B^c$.

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow \underbrace{x \in A \cap B}_{q}$, which contradicts our hypothesis $\underbrace{(A \cap B) = \emptyset}_{\sim q}$.

Since the assumption that $A \nsubseteq B^c$ leads to a contradiction, it must be false.

Hence, $A \subseteq B^c$. ■

Remark 5 *For problems 10 - 11, prove either one, by proving the contrapositive.*

10. $\underbrace{A \subseteq B}_{p} \Rightarrow \underbrace{(A \cap B) = A}_{q}$

Proof. We will prove the contrapositive, $\underbrace{(A \cap B) \neq A}_{\sim q} \Rightarrow \underbrace{A \nsubseteq B}_{\sim p}$.

Let the hypothesis be given. (i.e., Suppose that $(A \cap B) \neq A$).

$\Rightarrow$ either $(A \cap B) \nsubseteq A$ or $A \nsubseteq (A \cap B)$. (Otherwise, if each set were a subset of the other, the sets would be equal, contrary to our hypothesis.)

Since $(A \cap B) \subseteq A$ (always!) this leaves, as the only possibility, $A \nsubseteq (A \cap B)$.

$\Rightarrow \exists x \in A$ such that $x \notin (A \cap B)$

$\Rightarrow x \in A$, and either: $x \notin A$ or $x \notin B$.

i.e., $(x \in A) \wedge ((x \notin A) \vee (x \notin B))$

i.e., $\underbrace{x \in A \text{ and } x \notin A}_{\text{impossible}}$, or $x \in A$ and $x \notin B$

$\Rightarrow x \in A$ and $x \notin B$.

i.e., A contains an element that is not contained in B.

Hence, $A \nsubseteq B$.

We have shown that $(A \cap B) \neq A \Rightarrow A \nsubseteq B$. ■

11. $\underbrace{(A \cup B) \neq B}_{p} \Rightarrow \underbrace{A \not\subseteq B}_{q}$

Proof. We will prove the contrapositive, $\underbrace{A \subseteq B}_{\sim q} \Rightarrow \underbrace{(A \cup B) = B}_{\sim p}$

Let the hypothesis be given. (i.e., let $A \subseteq B$)

We must show that:

a. $(A \cup B) \subseteq B$

and

b. $B \subseteq (A \cup B)$

a. $\boxed{(A \cup B) \subseteq B}$

Let $x \in (A \cup B)$.

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in B$ (because $A \subseteq B$ by hypothesis.)

i.e., $x \in B$

We have shown that $x \in (A \cup B) \Rightarrow x \in B$.

Therefore, $(A \cup B) \subseteq B$

b. $\boxed{B \subseteq (A \cup B) \text{ (This is \textit{always} true.)}}$ ■

Remark 6 *Disprove both problems 12 and 13 by providing a counter-example.*

12. $(A \cup B)^c = A^c \cup B^c$

    Counter-example: $A = \{1, 2\}$; $B = \{2, 3\}$; $U = \{1, 2, 3\}$.

    Observe:

    $A \cup B = \{1, 2, 3\}$

    $(A \cup B)^c = \emptyset$

    $A^c = \{3\}$

    $B^c = \{1\}$

    $A^c \cup B^c = \{1, 3\}$

    $(A \cup B)^c = \emptyset \neq \{1, 3\} = A^c \cup B^c$

    i.e., $(A \cup B)^c \neq A^c \cup B^c$

    Alternatively: Consider another counter-example

    Counter-example: $A = \varnothing$; $B = U$

    Observe:

    $A^c = U$; $B^c = \varnothing$

    $A^c \cup B^c = U \cup \varnothing = U$

    $(A \cup B)^c = (\varnothing \cup U)^c = U^c = \varnothing$

    $A^c \cup B^c = U \neq \varnothing = (A \cup B)^c$

    i.e., $(A \cup B)^c \neq A^c \cup B^c$

13. $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$

For x and y, let's choose perfect squares. That will make the counter-example immediately obvious.

Let $x = 1$ and $y = 4$

$\sqrt{x+y} = \sqrt{1+4} = \sqrt{5} \neq 3 = \sqrt{1} + \sqrt{4} = \sqrt{x} + \sqrt{y}$

i.e., for $x = 1$ and $y = 4$, $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$

We can formulate an even better counter-example by letting x and y be the squares of the smaller two members of a "Pythagorean triple." (Where $x^2 + y^2$ is a perfect square.)

Let $x = 9$ and $y = 16$.

Then $\sqrt{x+y} = \sqrt{9+16} = 5 \neq 7 = \sqrt{9} + \sqrt{16} = \sqrt{x} + \sqrt{y}$

i.e., for $x = 9$ and $y = 16$, $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$