MTH 3311 Differential Equations Test #1 - Solutions

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Instructions. Show clearly how you arrive at your answers.

1. Classify the following according to **order** and **linearity.**

   (a) $y^{(3)} - x\left(y''\right)^3 - 5x^2y' + 6y = e^x$ **order 3, non-linear.**

   The highest order of derivative of y is 3. ($y^{(3)}$ is the *third derivative* of y – it is NOT y^3.) Since y'' is raised to a power other than 1, the equation is non-linear.

   (b) $y' = 2xy$ **order 1, linear.**

   The highest order of the derivative of y is 1. We can rewrite the equation in the form:

$$\underbrace{1 \cdot y'}_{a_1(x)y'} + \underbrace{(-2x)\,y}_{a_0(x)y} = \underbrace{0}_{F(x)}$$

   y and its derivatives are all raised to the 1st power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function.

   (c) $\frac{d^2y}{dx^2} + x^3\frac{dy}{dx} = 9x^2 + 3x$ **order 2, linear.**

   The highest order of the derivative of y is 2. We can rewrite the equation in the form:

$$\underbrace{1 \cdot \frac{d^2y}{dx^2}}_{a_2(x)y''} + \underbrace{x^3\frac{dy}{dx}}_{a_1(x)y'} + \underbrace{0 \cdot y}_{a_0(x)y} = \underbrace{9x^2 + 3x}_{F(x)}$$

   y and its derivatives are all raised to the 1st power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function.

   (d) $3y'' - y' - 10xy^3 = 10$ **order 2, non-linear.**

   The highest order of derivative of y is 2. (y'' is the *second derivative* of y.) Since y is raised to a power other than 1 $\left(y^3\right)$, the equation is non-linear.

(e) $3y''' - y'' - 10xy' = 10\sin(x)$ **order 3, linear.**

The highest order of the derivative of y is 3. We can rewrite the equation in the form:

$$\underbrace{3y'''}_{a_3(x)y'''} + \underbrace{(-1)\,y''}_{a_2(x)y''} + \underbrace{(-10x)\,\frac{dy}{dx}}_{a_1(x)y'} + \underbrace{0\cdot y}_{a_0(x)y} = \underbrace{\frac{10}{3}\sin(x)}_{F(x)}$$

y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function.

2. Solve: $\frac{dy}{dx} = x^2y$; Subject to the initial condition: $y(0) = 8$. (Assume $x, y \geq 0$).

The variables can be separated.

$\frac{dy}{dx} = x^2y \Rightarrow \frac{1}{y}dy = x^2dx$ Now Integrate!

$\int \frac{1}{y}dy = \int x^2dx \Rightarrow \ln(y) = \frac{1}{3}x^3 + C$

$\Rightarrow e^{\ln(y)} = e^{\frac{1}{3}x^3+C} \Rightarrow y = e^{\frac{1}{3}x^3}e^C \Rightarrow y = C_1e^{\frac{1}{3}x^3}$

Incorporating the initial condition, $y(0) = 8$, we have:

$8 = C_1e^{\frac{1}{3}(0)^3} = C_1$

i.e., $C_1 = 8$

Thus, our solution is $y = 8e^{\frac{1}{3}x^3}$

Alternative Solution is on the next page:

Alternative Solution:

Our differential equation can be written in the form: $\frac{dy}{dx} - x^2 y = 0 \Rightarrow \underbrace{\frac{dy}{dx}}_{\frac{dy}{dx}} + \underbrace{\left(-x^2\right) y}_{P(x)y} = \underbrace{0}_{Q(x)}$ This fits the form:

$$y' + P(x)\, y = Q(x)\,, \quad \text{with } P(x) = -x^2, \text{ and } Q(x) = 0$$

We can solve this using the "Integrating Factor Method"

1. Compute the integrating factor:

$e^{\int P(x)dx} = e^{\int -x^2 dx} = e^{-\frac{1}{3}x^3}$

2. Multiply both sides by the integrating factor

$e^{-\frac{1}{3}x^3} \frac{dy}{dx} - x^2 e^{-\frac{1}{3}x^3} y = 0 \cdot e^{-\frac{1}{3}x^3}$

$\underbrace{e^{-\frac{1}{3}x^3}}_{1^{\text{st}}} \underbrace{\frac{dy}{dx}}_{2^{\text{nd}} \text{ prime}} + \underbrace{\left(-x^2 e^{-\frac{1}{3}x^3}\right)}_{1^{\text{st}} \text{ prime}} \underbrace{y}_{2^{\text{nd}}} = 0$

3. Express the left hand side as the derivative of a product

$\Rightarrow \frac{d}{dx}\left[e^{-\frac{1}{3}x^3} y\right] = 0$

4. Integrate both sides w.r.t. x.

$\int \frac{d}{dx}\left[e^{-\frac{1}{3}x^3} y\right] dx = \int (0)\, dx \Rightarrow e^{-\frac{1}{3}x^3} y = C$

i.e., $e^{-\frac{1}{3}x^3} y = C$ Multiply both sides by $e^{\frac{1}{3}x^3}$

$\Rightarrow y = Ce^{\frac{1}{3}x^3}$

Incorporating the initial condition, $y(0) = 8$, we have:

$8 = Ce^{\frac{1}{3}(0)^3} = C$

i.e., $C = 8$

Thus, our solution is $y = 8e^{\frac{1}{3}x^3}$

3. Show that the function $y = c_1 e^{-3x} + c_2 e^{2x} + x^2$ is a solution of the differential equation $y'' + y' - 6y = -6x^2 + 2 + 2x$

Observe:

$$\begin{array}{|lcl|}\hline y &=& c_1 e^{-3x} + c_2 e^{2x} + x^2 \\ y' &=& -3c_1 e^{-3x} + 2c_2 e^{2x} + 2x \\ y'' &=& 9c_1 e^{-3x} + 4c_2 e^{2x} + 2 \\ \hline \end{array}$$

Plugging into the left side of the equation, we have:

$$\begin{array}{|lcl|}\hline y'' + y' - 6y &=& (9c_1 e^{-3x} + 4c_2 e^{2x} + 2) + (-3c_1 e^{-3x} + 2c_2 e^{2x} + 2x) - 6\left(c_1 e^{-3x} + c_2 e^{2x} + x^2\right) \\ &=& (9 + (-3) - 6)\, c_1 e^{-3x} + (4 + 2 - 6)\, c_2 e^{2x} + (2 + 2x - 6x^2) \\ &=& -6x^2 + 2x + 2 \\ \hline \end{array}$$

i.e., $y'' + y' - 6y = -6x^2 + 2x + 2$

Hence, $y = c_1 e^{-3x} + c_2 e^{2x} + x^2$ is a solution of the differential equation:

$y'' + y' - 6y = -6x^2 + 2x + 2.$

4. Solve: $y - x^2\frac{dy}{dx} = -xy; \quad y\left(1\right) = 1$. (Assume that $x, y > 0$)

The variables can be separated.

$y - x^2\frac{dy}{dx} = -xy \Rightarrow x^2\frac{dy}{dx} = xy + y \Rightarrow x^2\frac{dy}{dx} = \left(x+1\right)y \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{(x+1)}{x^2}$

$\Rightarrow \frac{1}{y}dy = \frac{(x+1)}{x^2}dx$ Now Integrate!

$\int \frac{1}{y}dy = \int \frac{(x+1)}{x^2}dx = \int \left(\frac{x}{x^2} + \frac{1}{x^2}\right)dx = \int \left(\frac{1}{x} + x^{-2}\right)dx$

i.e., $\int \frac{1}{y}dy = \int \left(\frac{1}{x} + x^{-2}\right)dx$

$\Rightarrow \ln\left(y\right) = \ln\left(x\right) - x^{-1} + C$

$\Rightarrow e^{\ln(y)} = e^{\ln(x) - x^{-1} + C} = e^{\ln(x)}e^{-x^{-1}}e^{C} = xe^{-x^{-1}}C_1 = C_1xe^{-\frac{1}{x}}$

i.e., $y = C_1xe^{-\frac{1}{x}}$
Incorporating the initial condition, $y\left(1\right) = 1$, we have:

$1 = C_1\left(1\right)e^{-\frac{1}{(1)}} = C_1e^{-1} = \frac{C_1}{e}$

i.e., $\frac{C_1}{e} = 1 \Rightarrow C_1 = e$

Thus, our solution is $y = exe^{-\frac{1}{x}}$

Alternative Solution is on the next page:

Alternative Solution: Our equation can be written in the form:

$y - x^2 \frac{dy}{dx} = -xy \Rightarrow x^2 \frac{dy}{dx} = xy + y \Rightarrow x^2 \frac{dy}{dx} = (x+1)\, y \Rightarrow \frac{dy}{dx} = \frac{(x+1)}{x^2} y$

$\Rightarrow \frac{dy}{dx} - \frac{(x+1)}{x^2} y = 0 \Rightarrow \underbrace{\frac{dy}{dx}}_{\frac{dy}{dx}} + \underbrace{\left(-\frac{(x+1)}{x^2}\right) y}_{P(x)y} = \underbrace{0}_{Q(x)}$

This fits the form:

$$y' + P(x)\, y = Q(x), \quad \text{with } P(x) = -\frac{(x+1)}{x^2}, \text{ and } Q(x) = 0$$

We can solve this using the "Integrating Factor Method"

1. Compute the integrating factor:

$e^{\int P(x)dx} = e^{\int -\frac{(x+1)}{x^2} dx} = e^{-\int \left(\frac{1}{x} + x^{-2}\right) dx} = e^{-\left(\ln(x) - x^{-1}\right) dx} = e^{-\ln(x) + x^{-1}} = e^{\ln\left(x^{-1}\right) + x^{-1}}$

$= e^{\ln\left(x^{-1}\right)} e^{x^{-1}} = x^{-1} e^{x^{-1}}$

2. Multiply both sides by the integrating factor

$\underbrace{x^{-1} e^{x^{-1}}}_{1^{\text{st}}} \underbrace{\frac{dy}{dx}}_{2^{\text{nd}} \text{ prime}} + \underbrace{\left(-x^{-1} e^{x^{-1}} \frac{(x+1)}{x^2}\right)}_{1^{\text{st}} \text{ prime}} \underbrace{y}_{2^{\text{nd}}} = 0$

3. Express the left hand side as the derivative of a product

$\Rightarrow \frac{d}{dx}\left[x^{-1} e^{x^{-1}} y\right] = 0$

4. Integrate both sides w.r.t. x.

$\int \frac{d}{dx}\left[x^{-1} e^{x^{-1}} y\right] dx = \int (0)\, dx \Rightarrow x^{-1} e^{x^{-1}} y = C$

i.e., $x^{-1} e^{x^{-1}} y = C$ Multiply both sides by $x e^{-x^{-1}}$

$\Rightarrow y = C x e^{-x^{-1}}$

Incorporating the initial condition, $y(1) = 1$, we have:

$1 = C(1)\, e^{-(1)^{-1}} = C e^{-1} = \frac{C}{e}$

i.e., $\frac{C}{e} = 1 \Rightarrow C = e$

Thus, our solution is $y = e x e^{-x^{-1}} = e x e^{-\frac{1}{x}}$

5. Solve: $\frac{1}{x^2}y' + 3y = 10$

It does NOT look like the variables can be separated.

But – it appears as though we can make it fit the form: $y' + P(x)\,y = Q(x)$

$\frac{1}{x^2}y' + 3y = 10 \Rightarrow \underbrace{y'}_{\frac{dy}{dx}} + \underbrace{3x^2 y}_{P(x)y} = \underbrace{10x^2}_{Q(x)}$

This fits the form:

$$y' + P(x)\,y = Q(x)\,, \quad \text{with } P(x) = 3x^2, \text{ and } Q(x) = 10x^2$$

We can solve this using the "Integrating Factor Method"

1. Compute the integrating factor:

$e^{\int P(x)dx} = e^{\int 3x^2 dx} = e^{x^3}$

2. Multiply both sides by the integrating factor

$e^{x^3}y' + 3x^2 e^{x^3} y = 10x^2 e^{x^3}$

$\underbrace{e^{x^3}}_{1^{\text{st}}} \underbrace{y'}_{2^{\text{nd}} \text{ prime}} + \underbrace{\left(3x^2 e^{x^3}\right)}_{1^{\text{st}} \text{ prime}} \underbrace{y}_{2^{\text{nd}}} = 10x^2 e^{x^3}$

3. Express the left hand side as the derivative of a product

$\Rightarrow \frac{d}{dx}\left[e^{x^3}y\right] = 10x^2 e^{x^3}$

4. Integrate both sides w.r.t. x.

$\int \frac{d}{dx}\left[e^{x^3}y\right]dx = \int 10x^2 e^{x^3} dx = 10\int \underbrace{e^{x^3}}_{e^u}\underbrace{x^2 dx}_{\frac{1}{3}du} = 10\int e^u \frac{1}{3}du = \frac{10}{3}\int e^u du = \frac{10}{3}e^u + C$

$= \frac{10}{3}e^{x^3} + C$

i.e., $e^{x^3}y = \frac{10}{3}e^{x^3} + C$ Multiply both sides by e^{-x^3}

$\Rightarrow y = \frac{10}{3} + Ce^{-x^3}$

Thus, our solution is $y = \frac{10}{3} + Ce^{-x^3}$

Alternative Solution is on the next page:

Alternative Solution: It turns out that we CAN separate the variables!

$\frac{1}{x^2}y' + 3y = 10$

$\Rightarrow \frac{1}{x^2}y' = 10 - 3y$

$\Rightarrow \frac{1}{x^2}\frac{dy}{dx} = 10 - 3y$

$\Rightarrow \frac{1}{10-3y}dy = x^2 dx$ Integrate!

$\int \frac{1}{10-3y}dy = \int x^2 dx$

Scratch Work:

$$\int \underbrace{\frac{1}{10-3y}}_{\frac{1}{u}} \underbrace{dy}_{-\frac{1}{3}du} = \int \frac{1}{u}\left(-\frac{1}{3}du\right) = -\frac{1}{3}\int \frac{1}{u}du = -\frac{1}{3}\ln(u) = -\frac{1}{3}\ln(10-3y)$$

Back to our integration:

$\int \frac{1}{10-3y}dy = \int x^2 dx$

$\Rightarrow -\frac{1}{3}\ln(10-3y) = \frac{1}{3}x^3 + C$

$\Rightarrow \ln(10-3y) = -x^3$

$\Rightarrow e^{\ln(10-3y)} = e^{-x^3+C}$

$\Rightarrow 10 - 3y = e^{-x^3}e^C$

$\Rightarrow 10 - 3y = C_1 e^{-x^3}$

$\Rightarrow -3y = -10 + C_1 e^{-x^3}$

$\Rightarrow y = \frac{10}{3} + C_2 e^{-x^3}$

Thus, our solution is $y = \frac{10}{3} + Ce^{-x^3}$

6. Determine whether or not the equation is exact. If the equation is exact, solve it.

$(1 + 3x^2 \sin(y))\, dx + (x^3 \cos(y))\, dy = 0$

$\underbrace{(1 + 3x^2 \sin(y))}_{M(x,y)} dx + \underbrace{(x^3 \cos(y))}_{N(x,y)} dy = 0$

By convention, we let $M(x, y)$ be the co-factor of dx and we let $N(x, y)$ be the co-factor of dy.

i.e., $M(x, y) = 1 + 3x^2 \sin(y)$ and $N(x, y) = x^3 \cos(y)$

If the Differential equation is **exact,** then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = 3x^2 \cos y = \frac{\partial N}{\partial x}$

Thus, the equation IS exact, and there exists a function $U(x, y)$ such that the equation $U(x, y) = C$ relates the solution y implicitly as a function of x.

To find $U(x, y)$, we compute the integrals $\int M(x, y)\, dx$ and $\int N(x, y)\, dy$.

$U(x, y) = \int M(x, y)\, dx = \int (1 + 3x^2 \sin(y))\, dx = x + x^3 \sin(y) + f(y) + C$

$U(x, y) = \int N(x, y)\, dy = \int (x^3 \cos(y))\, dy = x^3 \cos(y) + g(x) + C$

To find the unknown functions $f(y)$ and $g(x)$, we compare $\int M(x, y)\, dx$ and $\int N(x, y)\, dy$.

$$\begin{array}{ccccccccc} U(x,y) & = & x & + & x^3 \sin(y) & + & f(y) & + & C \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ U(x,y) & = & g(x) & + & x^3 \sin(y) & + & 0 & + & C \end{array}$$

Thus, $f(y) = 0$ and $g(x) = x$

Our solution $y = y(x)$ is given implicitly by the equation $U(x, y) = C$

$$\boxed{x + x^3 \sin(y) = C}$$

7. Solve: $(xy + y^2)\, dx - x^2 dy = 0$ Solve, using the substitution $v = \frac{y}{x}$

Re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$(xy + y^2)\, dx - x^2 dy = 0$

$\Rightarrow (xy + y^2) - x^2 \frac{dy}{dx} = 0$

$\Rightarrow \left(\frac{y}{x} + \frac{y^2}{x^2}\right) = \frac{dy}{dx}$

i.e., $\frac{dy}{dx} = \left(\frac{y}{x} + \left(\frac{y}{x}\right)^2\right)$ (i.e., $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$)

let $v = \frac{y}{x}$ (i.e., $y = vx$) $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting, we have:

$v + x\frac{dv}{dx} = v + v^2$ Now Separate!

$\Rightarrow x\frac{dv}{dx} = v^2$

$\Rightarrow v^{-2} dv = \frac{1}{x} dx$ Now Integrate!

$\int v^{-2} dv = \int \frac{1}{x} dx$

$\Rightarrow -v^{-1} = \ln|x| + C$

$\Rightarrow -\left(\frac{y}{x}\right)^{-1} = \ln|x| + C \Rightarrow -\frac{x}{y} = \ln|x| + C \Rightarrow -\frac{x}{\ln|x|+C} = y$

Our solution y is given by the equation:

$$y = -\frac{x}{\ln|x|+C}$$