Exercises Involving Real Numbers #4 - Solutions

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Instructions. Prove the following:

1. There exists a rational number between any two distinct real numbers.

   (i.e., If $x, y \in \mathbf{R}$ with $x \neq y$, then $\exists$ a rational number between x and y.)

   Proof. Let the hypothesis be given. (i.e., Let $x, y \in \mathbf{R}$ with $x \neq y$.)

   Without loss of generality, $x < y$

   $\Rightarrow \exists \varepsilon \in \mathbf{R}$ with $\varepsilon > 0$ such that $y - x = \varepsilon$.

   By the Axiom of Archimedes, $\exists N \in \mathbf{N}$ such that $N\varepsilon > 1$

   $\Rightarrow Ny - Nx = N\varepsilon > 1$

   i.e., $Ny - Nx > 1$

   $\Rightarrow \exists M \in \mathbf{Z}$ such that $Nx < M < Ny$

   $\Rightarrow x < \frac{M}{N} < y$

   Note that since $\frac{M}{N}$ is the quotient of integers (denominator non-zero), $\frac{M}{N} \in \mathbf{Q}$

   Hence, $\exists$ a rational number between x and y. ■

   Corollary 1 *There exists a rational number between any two distinct rational numbers.*

   Corollary 2 *There exists a rational number between any two distinct irrational numbers.*

2. There exists an irrational number between any two distinct real numbers.

   (i.e., If $x, y \in \mathbf{R}$ with $x \neq y$, then $\exists$ an irrational number between x and y.)

   Proof. Let the hypothesis be given. (i.e., Let $x, y \in \mathbf{R}$ with $x \neq y$.)

   Without loss of generality, $x < y$

   $\Rightarrow \exists \varepsilon \in \mathbf{R}$ with $\varepsilon > 0$ such that $y - x = \varepsilon$.

   By the Axiom of Archimedes, $\exists K \in \mathbf{Q}^c$ such that $K\varepsilon > 2$

   $\Rightarrow Ky - Kx = K\varepsilon > 2$

   i.e., $Ky - Kx > 2$

   $\Rightarrow \exists M \in \mathbf{Z}$ with $M \neq 0$ such that $Kx < M < Ky$

   $\Rightarrow x < \frac{M}{K} < y$

   Note that since $\frac{M}{K}$ is the quotient of a non-zero rational and an irrational, $\frac{M}{K} \in \mathbf{Q}^c$

   $\exists$ an irrational number between x and y.) ■

   Corollary 3 *There exists an irrational number between any two distinct rational numbers.*

   Corollary 4 *There exists an irrational number between any two distinct irrational numbers.*

3. Given a rational number z and any irrational number x, there exists an irrational number y such that $x + y = z$.

   Proof. Let the hypothesis be given. (i.e., Let $z \in \mathbf{Q}$ be given, and suppose that $x \in \mathbf{Q}^c$.)

   Let $y \in \mathbf{Q}^c$ be given by $y = z - x$.

   (Note that since y is the difference of an irrational and a rational, $y \in \mathbf{Q}^c$.)

   Observe: $x + y = x + (z - x) = z$.

   i.e., $x + y = z$ ■