Calc 2 Test 2 - Solutions

Spring 1987

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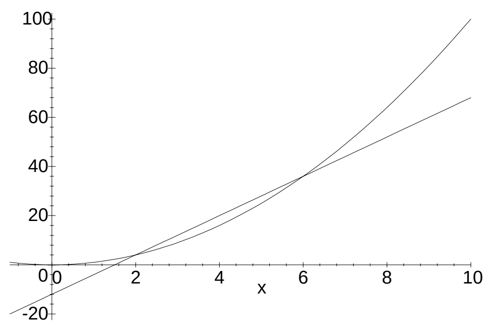
1. Compute: $\lim_{\triangle x \to 0} \Sigma_{i=1}^{n} \left(4x_i^2 - 3x_i + 2\right) \triangle x$,$a = 1, b = 2$

   $= \int_1^2 \left(4x^2 - 3x + 2\right) dx = \left[\frac{4}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_1^2$

   $= \left(\frac{32}{3} - 6 + 4\right) - \left(\frac{4}{3} - \frac{3}{2} + 2\right) = \frac{41}{6}$

2. Compute the area bounded by $y = x^2$ and $y = 8x - 12$

   (a) First, we must Graph the functions and find the points of intersection.



$y = x^2$ and $y = 8x - 12$

To find the points of intersection, set the y coordinates equal to one another.

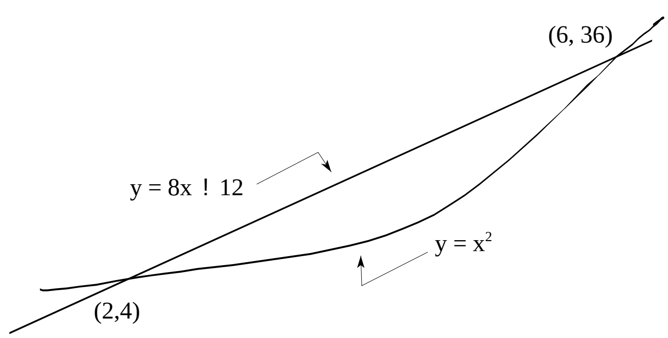
$y = x^2 = 8x - 12 \Rightarrow x^2 - 8x + 12 = 0 \Rightarrow (x - 2)(x - 6) = 0$

$\Rightarrow x = 2, 6$

$x = 2 \Rightarrow y = (2)^2 = 4$

$x = 6 \Rightarrow y = (6)^2 = 36$

Therefore, $(2, 4)$ and $(6, 36)$ are the points of intersection.



The area is the region bounded from $x = 2$ to $x = 6$. Over the interval $[2, 6]$,
$8x - 12 \geq x^2$.

So the area $= \int_2^6 ((8x - 12) - x^2)\, dx = \left[4x^2 - 12x - \frac{1}{3}x^3\right]_2^6$

$= \left[144 - 72 - \frac{216}{3}\right] - \left[16 - 24 - \frac{8}{3}\right]$

$= \frac{32}{3}$

3. Find the area bounded by $y = x^2 - 1$ and $y = x + 1$

(a) First, graph the functions, and find the points of intersection. To do this, we set the y coordinates equal to one another. $y = x^2 - 1 = x + 1$

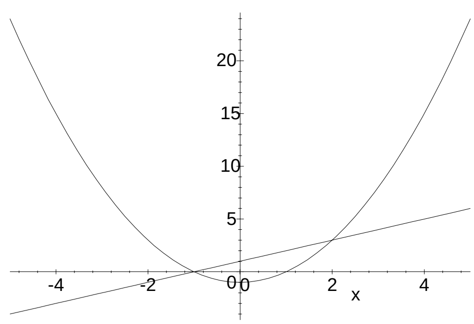
$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x + 1)(x - 2) = 0$

$\Rightarrow x = -1, 2$

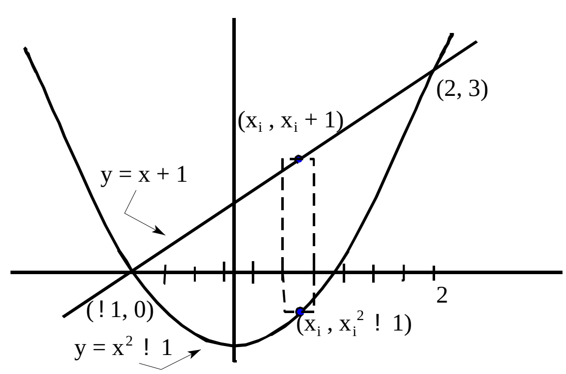
$$\left.\begin{array}{|l|}\hline x = -1 \Rightarrow y = (-1) + 1 = 0 \\ \hline x = 2 \Rightarrow y = (2) + 1 = 3 \\ \hline \end{array}\right\} \Rightarrow (-1, 0) \text{ and } (2, 3) \text{ are the points of}$$

intersection.



$y = x^2 - 1$ and $y = x + 1$

(b) Sketch a thin rectangle of width Δx, and partition the interval spanned by the region.



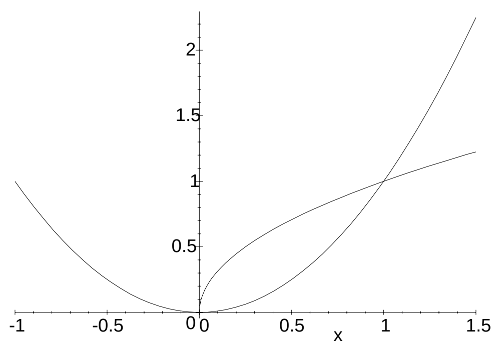
Area of the i^{th} rect $=\underbrace{\left[(x_i+1)-\left(x_i^2-1\right)\right]}_{\text{height}}\underbrace{\triangle x}_{\text{width}}=\left[x_i-x_i^2+2\right]\triangle x$

(c) Add up the areas of all of the rectangles to approximate the area of the region.
Area total $\approx \sum_{i=1}^{n}\left[x_i-x_i^2+2\right]\triangle x$

(d) Let $\triangle x\to 0$.
Area $=\lim_{\triangle x\to 0}\sum_{i=1}^{n}\left[x_i-x_i^2+2\right]\triangle x=\int_{-1}^{2}\left(x-x^2+2\right)dx$
$=\left[\frac{x^2}{2}-\frac{x^3}{3}+2x\right]_{-1}^{2}=\left[\left(\frac{2^2}{2}-\frac{2^3}{3}+2\,(2)\right)-\left(\frac{(-1)^2}{2}-\frac{(-1)^3}{3}+2\,(-1)\right)\right]$
$=\frac{9}{2}$

4. Use the Disc Method to compute the volume of the solid of revolution generated by revolving the region bounded by $y=x^{\frac{1}{2}}$, and $y=x^2$ about the axis $y=-1$.

(a) First, graph the functions, and find the points of intersection.

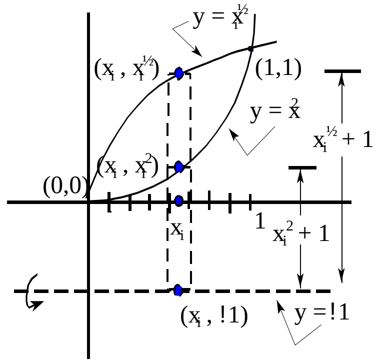


$y=x^{\frac{1}{2}}$ and $y=x^2$

To find the points of intersection, set the y coordinates equal to one another, and solve for x.

$y=x^{\frac{1}{2}}=x^2$	
$\Rightarrow x=x^4$	
$\Rightarrow x^4-x=0$	
$\Rightarrow x\left(x^3-1\right)=0$	
$\Rightarrow x=0$, or $x=1$	
points of intersection	$(0,0)$ and $(1,1)$

(b) Inscribe a rectangle perpendicular to the axis of revolution, partition the interval spanned by the region into sub-intervals.



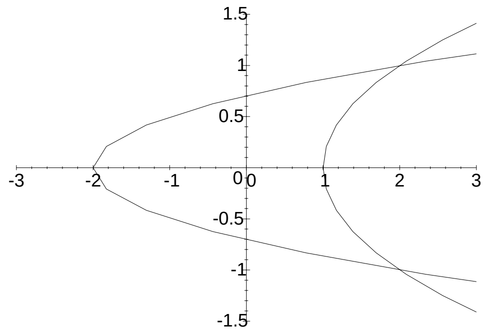
(c) Revolve the rectangle about the axis of revolution to form the ith disc.
Vol i^{th} Disc $= \pi R_i^2 \triangle x$
$= \pi \left(x_i^{\frac{1}{2}} + 1 \right)^2 \triangle x$
$= \pi \left(x_i + 2x_i^{\frac{1}{2}} + 1 \right) \triangle x$
Vol i^{th} hole $= \pi r_i^2 \triangle x$
$= \pi \left(x_i^2 + 1 \right)^2 \triangle x$
$= \pi \left(x_i^4 + 2x_i^2 + 1 \right) \triangle x$
Vol i^{th} washer $=$ Vol i^{th} Disc $-$Vol i^{th} hole $= \pi \left(x_i + 2x_i^{\frac{1}{2}} + 1 \right) \triangle x - \pi \left(x_i^4 + 2x_i^2 + 1 \right) \triangle x$
$= \pi \left(-x_i^4 - 2x_i^2 + x_i + 2x_i^{\frac{1}{2}} \right) \triangle x$

(d) Approximate the volume of the solid of revolution by adding up the volumes of the washers.
Vol solid $\approx \sum_{i=1}^{n} \pi \left(-x_i^4 - 2x_i^2 + x_i + 2x_i^{\frac{1}{2}} \right) \triangle x$

(e) Make this exact by letting $\triangle x \to 0$.
Vol solid $= \lim_{\triangle x \to 0} \sum_{i=1}^{n} \pi \left(-x_i^4 - 2x_i^2 + x_i + 2x_i^{\frac{1}{2}} \right) \triangle x$
$= \pi \int_{x=0}^{x=1} \left(-x^4 - 2x^2 + x + 2x^{\frac{1}{2}} \right) dx$
$= \pi \left[-\frac{x^5}{5} - \frac{2}{3}x^3 + \frac{x^2}{2} + \frac{4}{3}x^{\frac{3}{2}} \right]_{x=0}^{x=1}$
$= \pi \left[-\frac{1}{5} - \frac{2}{3} + \frac{1}{2} + \frac{4}{3} \right] - \pi [0]$
$= \frac{29\pi}{30}$

5. Use the Shell Method to compute the volume of the solid of revolution generated by revolving the region bounded by the graphs $x = 4y^2 - 2$, $x = y^2 + 1$, and the y -axis about the x-axis.

(a) First, graph the functions and find the points of intersection.



$$x = 4y^2 - 2 \text{ and } x = y^2 + 1$$

To find the points of intersection, set the x coordinates equal to one another, and solve for y.

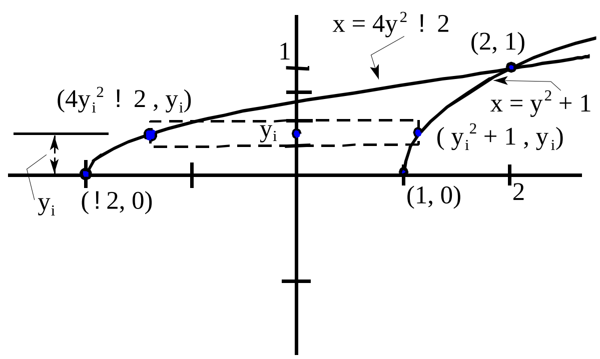
$x = 4y^2 - 2 = y^2 + 1$

$\Rightarrow 3y^2 - 3 = 0 \Rightarrow y^2 - 1 = 0$

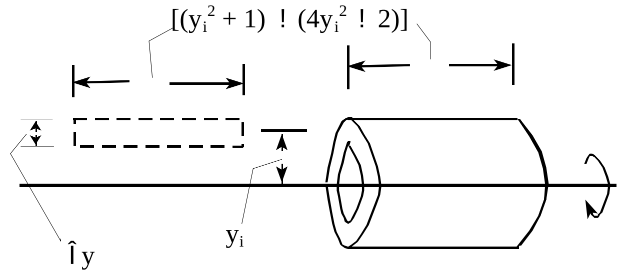
$\Rightarrow (y+1)(y-1) = 0 \Rightarrow y = \pm 1$

$\left.\begin{array}{|l|}\hline y = 1 \Rightarrow x = 1^2 + 1 = 2 \\ \hline y = -1 \Rightarrow x = (-1)^2 + 1 - 2 \\ \hline \end{array}\right\} \Rightarrow (2,1)$ and $(2,-1)$ are the points of intersection.

(b) Sketch a rectangle parallel to the axis of revolution and partition the interval that is spanned by the region.



(c) Revolve the ith rectangle about the axis of revolution to form the ith shell. Then compute the volume of the ith shell.



Vol i^{th} shell $= 2\pi r h \triangle y = 2\pi (y_i) [(y_i^2+1) - (4y_i^2-2)] \triangle y$
$= 2\pi (y_i)(-3y_i^2+3) \triangle y = 2\pi (-3y_i^3 + 3y_i) \triangle y$

(d) Approximate the volume of the solid of revolution by adding the volumes of the shells.
Vol. of solid $\approx \sum_{i=1}^{n} 2\pi (-3y_i^3+3y_i) \triangle y$

(e) Let $\triangle x \to 0$ to get the exact volume.
Vol Solid $= \lim_{\triangle x \to 0} \sum_{i=1}^{n} 2\pi (-3y_i^3+3y_i) \triangle y = 2\pi \int_{y=0}^{y=1} (-3y^3+3y)\, dy$
$= 2\pi \left[-\frac{3}{4}y^4 + \frac{3}{2}y^2\right]_0^1 = 2\pi \left[\left(-\frac{3}{4}+\frac{3}{2}\right) - (0+0)\right]$
$= 2\pi \left(\frac{3}{4}\right) = \frac{3\pi}{2}$

6. Compute: $\int_0^1 \sqrt{5x+4}dx$

Re-write as: $\int_0^1 (5x+4)^{\frac{1}{2}}\, dx$

1. Is u-sub appropriate?
   1. Composite Function? yes!
      $(5x+4)^{\frac{1}{2}}$
      Let $u = 5x+4$
   2. Approx Funct/deriv pair? yes!
      $\underbrace{5x+4}_{\text{funct}} \to \underbrace{1}_{\text{deriv}}$
      Let $u = 5x+4$
2. Compute du

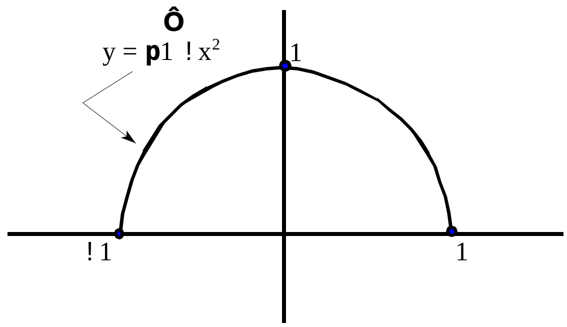
$u = 5x+4$
$du = 5dx$
$\frac{1}{5}du = dx$
When $x = 0$; $u = 5x+4 = 5(0)+4 = 4$
i.e. when $x = 0$; $u = 4$
When $x = 1$; $u = 5x+4 = 5(1)+4 = 9$
i.e. when $x = 1$; $u = 9$

3. Analyze the integral in terms of u and du.
   $\int_{x=0}^{x=1} \underbrace{(5x+4)^{\frac{1}{2}}}_{u^{\frac{1}{2}}} \underbrace{dx}_{\frac{1}{5}du} = \int_{u=4}^{u=9} u^{\frac{1}{2}} \cdot \frac{1}{5}du = \frac{1}{5}\int_{u=4}^{u=9} u^{\frac{1}{2}} du$

4. Integrate:
$\frac{1}{5}\int_{u=4}^{u=9} u^{\frac{1}{2}}du = \frac{1}{5}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{u=4}^{u=9} = \frac{2}{15}\left[u^{\frac{3}{2}}\right]_{u=4}^{u=9} = \frac{2}{15}\left((9)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right)$
$= \frac{2}{15}(27-8) = \frac{38}{15}$

7. Use geometry to compute: $\int_{-1}^{1}\sqrt{1-x^2}dx$

First, observe that $y=\sqrt{1-x^2}$ is the equation of the semi-circle of radius 1, centered at the origin, and lying above the x-axis.



$$y=\sqrt{1-x^2}$$

A circle of radius 1 has an area of $\pi r^2 = \pi \cdot 1^2 = \pi$.

The integral we seek will give us the area of half a circle of radius 1, because the graph of $y=\sqrt{1-x^2}$ is a semi-circle which lies above the x-axis.
Therefore, $\int_{-1}^{1}\sqrt{1-x^2}dx = \frac{1}{2}\pi$

8. Suppose that $\int_0^1 (f(x) - g(x))\,dx = 3$ and $\int_0^1 g(x)\,dx = 1$.

Compute: $\int_0^1 3f(x)\,dx$

Observe: $3 = \int_0^1 (f(x) - g(x))\,dx = \int_0^1 f(x)\,dx - \int_0^1 g(x)\,dx = \int_0^1 f(x)\,dx - 1$

i.e., $3 = \int_0^1 f(x)\,dx - 1 \Rightarrow \int_0^1 f(x)\,dx = 4$

$\Rightarrow \int_0^1 3f(x)\,dx = 3\int_0^1 f(x)\,dx = 3\cdot 4 = 12$

9. Suppose that $\int_0^1 (f(x) + 2g(x))\,dx = 8$ and $\int_0^1 (2f(x) - g(x))\,dx = -2$.

Compute: $\int_0^1 f(x)\,dx$

Observe: $\int_0^1 (f(x) + 2g(x))\,dx + 2\cdot\int_0^1 (2f(x) - g(x))\,dx$

$= \int_0^1 (f(x) + 2g(x))\,dx + \int_0^1 (4f(x) - 2g(x))\,dx$

$= \int_0^1 f(x)\,dx + \int_0^1 2g(x)\,dx + \int_0^1 4f(x)\,dx - \int_0^1 2g(x)\,dx$

$= \int_0^1 f(x)\,dx + \int_0^1 4f(x)\,dx = \int_0^1 f(x)\,dx + 4\int_0^1 f(x)\,dx = 5\int_0^1 f(x)\,dx$

i.e., $\int_0^1 (f(x) + 2g(x))\,dx + 2\cdot\int_0^1 (2f(x) - g(x))\,dx = 5\int_0^1 f(x)\,dx$

$\Rightarrow 8 + 2(-2) = 5\int_0^1 f(x)\,dx$

$\Rightarrow 4 = 5\int_0^1 f(x)\,dx$

$\Rightarrow \int_0^1 f(x)\,dx = \frac{4}{5}$

10. Suppose that $\int_0^4 g(x)\,dx = -3$ and $\int_4^2 3g(x)\,dx = -2$.

Compute: $\int_0^2 2g(x)\,dx$

Observe: $\int_2^4 3g(x)\,dx = -\int_4^2 3g(x)\,dx = -(-2) = 2$

$\Rightarrow \int_2^4 3g(x)\,dx = 2$

$\Rightarrow 3\int_2^4 g(x)\,dx = 2 \Rightarrow \int_2^4 g(x)\,dx = \frac{2}{3}$

Next, note that $\int_0^2 g(x)\,dx + \int_2^4 g(x)\,dx = \int_0^4 g(x)\,dx = -3$

i.e., $\int_0^2 g(x)\,dx + \int_2^4 g(x)\,dx - -3$

$\Rightarrow \int_0^2 g(x)\,dx = -3 - \int_2^4 g(x)\,dx = -3 - \frac{2}{3} = -\frac{11}{3}$

i.e., $\int_0^2 g(x)\,dx = -\frac{11}{3}$

$\Rightarrow \int_0^2 2g(x)\,dx = 2\int_0^2 g(x)\,dx = 2 \cdot -\frac{11}{3} = -\frac{22}{3}$