MTH 4441 Homework #2 Groups - Solutions

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For Exercises 1-10, decide whether each of the given sets is a group with respect to the given operation. If it is NOT a group, state at least one of the group axioms that fails to hold.

Group Axioms for $(G, *)$

- The Binary Operator $*$ is **closed** on G.
- $*$ is associative
- $(G, *)$ has an identity element
- Each element $x \in G$ has an inverse.

1. The set $\mathbb{Z}^+$ of all positive integers with operation addition.

   $(\mathbb{Z}^+, +)$ is **NOT a group.** $(\mathbb{Z}^+, +)$ has no additive identity. (The additive identity would be 0, but $0 \notin \mathbb{Z}$.)

   Also: The additive inverse of each element $n \in \mathbb{Z}^+$ is the *negative* integer $-n$, which is NOT an element of $\mathbb{Z}^+$.

2. The set $\mathbb{Z}^+$ of all positive integers with operation multiplication.

   $(\mathbb{Z}^+, \cdot)$ is **NOT a group.** The multiplicative inverse of each element $n \in \mathbb{Z}^+$ is the rational number $\frac{1}{n}$, which is NOT an element of $\mathbb{Z}^+$ for $n > 1$.

3. The set $\mathbb{Q}$ of all rational numbers with operation addition.

   $(\mathbb{Q}, +)$ **IS a group.**

   The operation $+$ is closed on $\mathbb{Q}$, since the sum of two rational numbers is also a rational number.

   $0 \in \mathbb{Q}$ is the additive identity.

   Given $\frac{m}{n} \in \mathbb{Q}$, the element $-\frac{m}{n} \in \mathbb{Q}$ is the additive inverse.

   The operation $+$ is associative (We know this because The operation $+$ is associative for ALL real numbers.)

4. The set $\mathbb{Q}'$ of all irrational numbers with operation addition.

   $(\mathbb{Q}', +)$ is **NOT a group.**

   The operation $+$ is **not closed** on $\mathbb{Q}'$. (To see this, observe that the sum of irrational numbers $0.101101110111110\ldots$ and $0.01001000100001\ldots$ is the rational number $0.11111111111111\ldots$)

   Also: Since the *rational* number 0 is the additive identity for ALL subsets of the Real Numbers under addition, $(\mathbb{Q}', +)$ has **no additive identity,** since $0 \notin \mathbb{Q}'$

5. The set of all positive irrational numbers with operation multiplication.

   $\left((\mathbb{Q}')^+, \cdot \right)$ is **NOT a group.**

   The operation $\cdot$ is **not closed** on $\mathbb{Q}'$. (To see this, observe that the product of irrational numbers $\sqrt{2}$ and $\sqrt{2}$ is the rational number 2)

   Also: Since the *rational* number 1 is the multiplicative identity for ALL subsets of the Real Numbers under multiplication, , $\left((\mathbb{Q}')^+, \cdot \right)$ has **no additive identity,** since $1 \notin \mathbb{Q}'$**.**

6. The set $\mathbb{Q}^+$ of all positive rational numbers with operation multiplication.

   $(\mathbb{Q}^+, \cdot)$ **IS a group.**

   The operation $\cdot$ is closed on $\mathbb{Q}^+$, since the product of two positive rational numbers is also a positive rational number.

   $1 \in \mathbb{Q}^+$ is the multiplicative identity.

   Given $\frac{m}{n} \in \mathbb{Q}^+$, the element $\frac{n}{m} \in \mathbb{Q}^+$ is the multiplicative inverse.

   The operation $\cdot$ is associative (We know this because The operation $\cdot$ is associative for ALL real numbers.)

7. The set $\mathbf{E}$ of all even integers with operation addition.

   $(\mathbf{E}, +)$ **IS a group.**

   The operation $+$ is closed on $\mathbf{E}$, since the sum of two even numbers is also an even number.

   $0 \in \mathbf{E}$ is the additive identity

   Given the even number $2n \in \mathbf{E}$, the even number $-2n \in \mathbf{E}$ is the additive inverse.

   The operation $+$ is associative (We know this because The operation $+$ is associative for ALL real numbers.)

8. The set $\mathbf{E}$ of all even integers with operation multiplication.

   $(\mathbf{E}, \cdot)$ **is NOT a group.**

   The operation $\cdot$ IS closed on $\mathbf{E}$, since the product of two even numbers is also an even number.

   HOWEVER, the multiplicative identity $1 \notin \mathbf{E}$.

   ALSO, given the element $2n \in \mathbf{E}$, the multiplicative inverse, $\frac{1}{2n} \notin \mathbf{E}$.

9. The set of all multiples of 5 with operation addition.

   The set is denoted $5\mathbb{Z} = \{0, \pm 5, \pm 10, \pm 15, \ldots\}$

   $(5\mathbb{Z}, +)$ **IS a group.**

   $+$ is closed on $5\mathbb{Z}$. To see this, observe that given two elements $5j$ and $5k$ in $5\mathbb{Z}$, their sum $5j + 5k = 5(j + k) \in 5\mathbb{Z}$

   The element $0 = 5 \cdot 0 \in 5\mathbb{Z}$ is the additive identity.

   Given $5k \in 5\mathbb{Z}$, the element $-5k \in 5\mathbb{Z}$ is the additive inverse.

   The operation $+$ is associative (We know this because The operation $+$ is associative for ALL real numbers.)

10. The set of all multiples of 5 with operation multiplication.

    $(5\mathbb{Z}, \cdot)$ **is NOT a group.**

    $\cdot$ is closed on $5\mathbb{Z}$. To see this, observe that given two elements $5j$ and $5k$ in $5\mathbb{Z}$, their product $(5j)(5k) = 25jk = 5(5jk) \in 5\mathbb{Z}$

    HOWEVER, the multiplicative identity $1 \notin 5\mathbb{Z}$..

    ALSO, given $n \in 5\mathbb{Z}$, the multiplicative inverse $\frac{1}{n} \notin 5\mathbb{Z}$.

In Exercises 11-12, the given table defines an operation of multiplication on the set $S = \{e, a, b, c\}$. In each case, find a group axiom that fails to hold, and thereby show that S is **not** a group.

11.

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	b	a	b
b	b	c	b	c
c	c	e	c	e

Here are a few things:

Notice that the identity element e does not appear in the row headed by a. This means that a does not have a right inverse.

Notice that the identity element e does not appear in the row or column headed by b. This means that b has neither a right inverse nor a left inverse.

Notice that the identity e appears twice in the row headed by c – once in the column headed by a and once in the column headed by c. This means that both a and c are right inverses of c, violating the fact that an inverse is unique.

12.

$\cdot$	e	a	b	c
e	e	a	b	c
a	e	a	b	c
b	e	a	b	c
c	e	a	b	c

Here are a few things:

All entries in the column headed by e show that e is NOT a right identity. (i.e., $xe \neq x$ for any element except $x = e$.) So, **there is NO two-sided identity.**

The fact that $xy = y$ for all elements $x, y \in S$, tell us that each element $x \in S$ is a left identity, contradicting the fact that such an identity should be unique.

The facts that:

$xa \neq e, \forall x \in S$

$xb \neq e, \forall x \in S$

$xc \neq e, \forall x \in S$

tell us that none of the elements a, b, c have a left inverse.

In exercises, 13-18, let the binary operation be defined on $\mathbb{Z}$ by the rule given. Determine in each case whether $(\mathbb{Z}, *)$ is a group. If it is a group, determine if it is an abelian group. If it is NOT a group, state which conditions, if any fail to hold.

13. $x * y = x + y + 1$

This IS a group.

$*$ is closed on $\mathbb{Z}$

Find the identity: We want $e \in \mathbb{Z}$ such that $x * e = x + e + 1 = x$

$\Rightarrow x + e + 1 = x \Rightarrow e + 1 = 0 \Rightarrow e = -1$

Check: $x * e = x + e + 1 = x + (-1) + 1 = x$

Also: $e * x = e + x + 1 = (-1) + x + 1 = x$

i.e. For $e = -1$, we have: $e * x = x = x * e$ (i.e. $e = -1$ IS the identity)

Find the inverse: (i.e., For $x \in \mathbb{Z}$, find x^{-1})

We want x^{-1} such that $x * x^{-1} = x + x^{-1} + 1 = e = -1$

$\Rightarrow x + x^{-1} + 1 = -1 \Rightarrow x + x^{-1} = -2 \Rightarrow x^{-1} = -2 - x$

Check: $x * x^{-1} = x + x^{-1} + 1 = x + (-2 - x) + 1 = -1 = e$

Also: $x^{-1} * x = x^{-1} + x + 1 = (-2 - x) + x + 1 = -1 = e$

i.e., $x * x^{-1} = e = x * x^{-1}$ (i.e., Given $x \in \mathbb{Z}$, $x^{-1} = -2 - x$)

Regarding **associativity:**

$(x * y) * z = (x + y + 1) * z = (x + y + 1) + z + 1 = x + y + z + 1 + 1 = x + (y + z + 1) + 1$

$= x * (y + z + 1) = x * (y * z)$

i.e., $(x * y) * z = x * (y * z)$ (* is associative)

Furthermore, $(\mathbb{Z}, *)$ **IS an abelian group.**

This will follow, if we can show that $x * y = y * x$

Observe: $x * y = (x + y + 1) = (y + x + 1) = y * x$

14. $x * y = x + y - 1$

This IS a group.

$*$ is closed on $\mathbb{Z}$

$e = 1$ is the identity

Find the identity: We want $e \in \mathbb{Z}$ such that $x * e = x + e - 1 = x$

$\Rightarrow x + e - 1 = x \Rightarrow e - 1 = 0 \Rightarrow e = 1$

Check: $x * e = x + e - 1 = x + (1) - 1 = x$

Also: $e * x = e + x - 1 = (1) + x - 1 = x$

i.e. For $e = 1$, we have: $e * x = x = x * e$ (i.e. $e = 1$ IS the identity)

Find the inverse: (i.e., For $x \in \mathbb{Z}$, find x^{-1})

We want x^{-1} such that $x * x^{-1} = x + x^{-1} - 1 = e = 1$

$\Rightarrow x + x^{-1} - 1 = 1 \Rightarrow x + x^{-1} = 2 \Rightarrow x^{-1} = 2 - x$

Check: $x * x^{-1} = x + x^{-1} - 1 = x + (2 - x) - 1 = 1 = e$

Also: $x^{-1} * x = x^{-1} + x - 1 = (2 - x) + x - 1 = 1 = e$

i.e., $x * x^{-1} = e = x * x^{-1}$ (i.e., Given $x \in \mathbb{Z}$, $x^{-1} = 2 - x$)

Regarding **Associativity:**

$(x * y) * z = (x + y - 1) * z = (x + y - 1) + z - 1 = x + y + z - 1 - 1 = x + (y + z - 1) - 1$

$$= x * (y + z - 1) = x * (y * z)$$

i.e., $(x * y) * z = x * (y * z)$ (* is associative)

Furthermore, $(\mathbb{Z}, *)$ **IS an abelian group.**

This will follow, if we can show that $x * y = y * x$

Observe: $x * y = (x + y - 1) = (y + x - 1) = y * x$

15. $x * y = x + xy$

This is NOT a group.

$*$ is closed on $\mathbb{Z}$

Is there an identity? (i.e., is there an element e such that $e * x = x = x * e$?)

Observe: $x * e = x + xe = x, \Rightarrow xe = 0$

$\Rightarrow xe = 0 \Rightarrow e = 0$

i.e., $x * 0 = x,$ so $e = 0$ may be the identity.

Let's check to see if $0 * x = x$.

$0 * x = 0 + 0 \cdot x = 0.$

i.e., $x * 0 = x$, but $0 * x = 0$.

So $e = 0$ is NOT a (two-sided) identity

i.e., **There is no identity**

Hence, given $x \in \mathbb{Z}$, there can be no x^{-1}.

Given $x \in \mathbb{Z}$, x^{-1} **Does Not Exist)**

Regarding Associativity:

$(x * y) * z = (x + xy) * z = (x + xy) + (x + xy)\, z = x + xy + xz + xyz$

$x * (y * z) = x * (y + yz) = x + x\,(y + yz) = x + xy + xyz$

i.e., $(x * y) * z \neq x * (y * z)$ (*** is NOT associative**)

16. $x * y = xy + y$

This is NOT a group.

$*$ is closed on $\mathbb{Z}$

Is there an identity? (i.e., is there an element e such that $e * x = x = x * e$?)

Observe: $x * e = xe + e = x \Rightarrow (x+1)\,e = x \Rightarrow e = \frac{x}{x+1}$

i.e., $e = \frac{x}{x+1}$

Note that e is NOT a constant value - it's value depends on the value of x. So there is no single element e such that $x * e = x$.

Therefore, **there is no identity**

Hence, given $x \in \mathbb{Z}$, there can be no x^{-1}.

Given $x \in \mathbb{Z}$, x^{-1} **Does Not Exist)**

Regarding Associativity:

$(x * y) * z = (xy + y) * z = (xy + y)\,z + z = xyz + yz + z$

$x * (y * z) = x * (yz + z) = x\,(yz + z) + (yz + z) = xyz + xz + yz + z$

i.e., $(x * y) * z \neq x * (y * z)$ (*** is NOT associative**)

17. $x * y = x + xy + y$

This is NOT a group.

$*$ is closed on $\mathbb{Z}$

Is there an identity? (i.e., is there an element e such that $e * x = x = x * e$?)

Observe: $x * e = x + xe + e = x \Rightarrow xe + e = 0 \Rightarrow (x+1)\,e = 0 \Rightarrow e = \frac{0}{x+1}$

i.e., $e = 0$

Also:
$e * x = e + ex + x = x \Rightarrow e + ex = 0 \Rightarrow e\,(1+x) = 0 \Rightarrow e = \frac{0}{1+x}$

i.e., $e = 0$

Next, $\forall x \in \mathbb{Z}$, does there exist an x^{-1} ?

Consider:
$x * x^{-1} = x + xx^{-1} + x^{-1} = 0 \Rightarrow xx^{-1} + x^{-1} = -x \Rightarrow (x+1)\,x^{-1} = -x$

$\Rightarrow x^{-1} = \frac{-x}{x+1}$
Note that a consequence of this result is that x^{-1} is undefined for $x = -1$. (i.e., $x = -1$ has no inverse.)

x **inverse does not exist for every element in** $\mathbb{Z}$

Regarding Associativity:

$(x * y) * z = (x + xy + y) * z = (x + xy + y) + (x + xy + y)\,z + z = x + xy + y + xz + xyz + yz + z$

$= x + y + z + xy + xz + yz + xyz$

$x * (y * z) = x * (y + yz + z) = x + x\,(y + yz + z) + (y + yz + z) = x + xy + xyz + xz + y + yz + z$

$= x + y + z + xy + xz + yz + xyz$

i.e., $(x * y) * z = x * (y * z)$ (*** IS associative**)

18. $x * y = x - y$

This is NOT a group.

$*$ is closed on $\mathbb{Z}$

Is there an identity? (i.e., is there an element e such that $e * x = x = x * e$?)

Observe: $x * e = x - e = x \Rightarrow -e = 0 \Rightarrow e = 0$

i.e., $e = 0$

Also:
$e * x = e - x = x \Rightarrow e = 2x$

i.e., $e = 2x$

The left and right sided identities are not equal, so there is **NO Identity.**

Consequently, $\forall x \in \mathbb{Z}$, **there does NOT exist an inverse.**

Regarding Associativity:

$(x * y) * z = (x + xy + y) * z = (x + xy + y) + (x + xy + y)\, z + z = xz + xyz + yz + z$

$x * (y * z) = x * (y + yz + z) = x + x\,(y + yz + z) + (y + yz + z) = x + xy + xyz + y + yz + z$

i.e., $(x * y) * z \neq x * (y * z)$ (*** is NOT associative**)

In exercises, 19-21, Fill in the group table for $(G, *)$ in as many different ways as possible.

19.

$*$	e	a
e		
a		

$*$	e	a
e	e	a
a	a	e

This is the only possibility.

20.

$*$	e	a	b
e			
a			
b			

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

This is the only possibility.

21.

$*$	e	a	b	c
e				
a				
b				
c				

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$*$	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	a	c
c	c	b	c	e

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

are all possibilities.