MTH 3311 Test #3

SPRING 2021

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Show CLEARLY how you arrive at your answers.

1. Solve the Differential Equation: $y'' - 2y' - 8y = -40\cos(x) - 10\sin(x)$

   First, we must find the complementary solution, y_c

   We consider the complementary equation $y'' - 2y' - 8y = 0$

   Using differential operator notation, this becomes:

   $D^2y - 2Dy - 8y = 0$

   $\Rightarrow \underbrace{(D^2 - 2D - 8)}_{\phi(D)} y = 0$

   $\Rightarrow \phi(D) = D^2 - 2D - 8$

   We find the roots of the auxilliary equation: $\phi(m) = 0$

   $\Rightarrow \underbrace{m^2 - 2m - 8}_{\phi(m)} = 0$

   $\Rightarrow (m + 2)(m - 4) = 0$

   $\Rightarrow m_1 = -2$ and $m_2 = 4$

   Our Complementary Solution is $y_c = c_1e^{m_1x} + c_2e^{m_2x}$

   $\Rightarrow y_c = c_1e^{-2x} + c_2e^{4x}$

   Next, we find a particular solution y_p to the equation:

   $y'' - 2y' - 8y = -40\cos(x) - 10\sin(x)$

   Since the right hand side is a linear combination of sines and cosines, we guess that the particular solution is of the form:

   $y = c_1\cos(x) + c_2\sin(x)$

$$\left.\begin{array}{rrcl} & y & = & c_1\cos(x) + c_2\sin(x) \\ \Rightarrow & y' & = & -c_1\sin(x) + c_2\cos(x) \\ \Rightarrow & y'' & = & -c_1\cos(x) - c_2\sin(x) \end{array}\right\} \begin{array}{l} \text{we plug these into the equation:} \\ y'' - 2y' - 8y = -40\cos(x) - 10\sin(x) \end{array}$$

$$\begin{array}{rcrcrcl}
y'' & = & -c_1\cos(x) & & -c_2\sin(x) & & \\
-2y' & = & -2c_2\cos(x) & & +2c_1\sin(x) & & \\
-8y & = & -8c_1\cos(x) & & -8c_2\sin(x) & & \\
\hline
y''-2y'-8y & = & (-c_1-2c_2-8c_1)\cos(x) & + & (-c_2+2c_1-8c_2)\sin(x) & = & -40\cos(x)-10\sin(x)
\end{array}$$

i.e., $(-c_1 - 2c_2 - 8c_1)\cos(x) + (-c_2 + 2c_1 - 8c_2)\sin(x) = -40\cos(x) - 10\sin(x)$

$\Rightarrow (-9c_1 - 2c_2)\cos(x) + (2c_1 - 9c_2)\sin(x) = -40\cos(x) - 10\sin(x)$

Equating the coefficients of $\sin(x)$ and $\cos(x)$ on both sides of the equation, we have:

$-9c_1 - 2c_2 = -40$ and $2c_1 - 9c_2 = -10$

Solving these equations simultaneously, we have:

$$\begin{array}{rcrcr}
-9c_1 & - & 2c_2 & = & -40 \\
2c_1 & - & 9c_2 & = & -10 \\
\hline
\end{array}
\quad \Rightarrow \quad
\begin{array}{rcrcr}
-18c_1 & - & 4c_2 & = & -80 \\
18c_1 & - & 81c_2 & = & -90 \\
\hline
 & - & 85c_2 & = & -170
\end{array}$$

i.e., $-85c_2 = -170 \Rightarrow c_2 = 2$

Plugging this value of c_2 into the equation $2c_1 - 9c_2 = -10$ yields: $2c_1 - 9(2) = -10$

$\Rightarrow 2c_1 = 8$

$\Rightarrow c_1 = 4$

Thus, our particular solution $y = c_1\cos(x) + c_2\sin(x)$ becomes:

$y_p = 4\cos(x) + 2\sin(x)$

Our General Solution is $y_g = \underbrace{4\cos(x) + 2\sin(x)}_{y_p} + \underbrace{c_1 e^{-2x} + c_2 e^{4x}}_{y_c}$

2. Solve the Differential Equation: $x^2y'' - 4xy' + 4y = 3x^2 + 4$

First, find the solution to the complementary equation $x^2y'' - 4xy' + 4y = 0$

Our strategy is to seek solutions of the form:

$$y = x^\lambda$$

$$\Rightarrow y' = \lambda x^{\lambda-1}$$

$$\Rightarrow y'' = \lambda(\lambda - 1)\, x^{\lambda-2} = \left(\lambda^2 - \lambda\right) x^{\lambda-2}$$

Plugging these into the complementary equation $x^2y'' - 4xy' + 4y = 0$, we have:

$$x^2\left(\lambda^2 - \lambda\right) x^{\lambda-2} - 4x\lambda x^{\lambda-1} + 4x^\lambda = 0$$

$$\Rightarrow \left(\lambda^2 - \lambda\right) x^\lambda - 4\lambda x^\lambda + 4x^\lambda = 0$$

$$\Rightarrow \left(\lambda^2 - \lambda\right) - 4\lambda + 4 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 4) = 0$$

$$\Rightarrow \lambda_1 = 1;\ \lambda_2 = 4$$

Our complementary solution is:

$$y_c = c_1x^{\lambda_1} + c_2x^{\lambda_2} = c_1x^1 + c_2x^4$$

(Continued)

Next, we find our particular solution

Since the right hand side of the equation is the polynomial $3x^2+4$, we guess that the particular solution is a polynomial having only terms of the same degree that appear on the right hand side of the original equation.

Thus, we guess that:

$$y = Ax^2 + C$$

$\Rightarrow y' = 2Ax$

$\Rightarrow y'' = 2A$

To find A, B and C, we plug these into the original equation, $x^2y'' - 4xy' + 4y = 3x^2 + 4$.

This yields:

$$x^2(2A) - 4x(2Ax) + 4(Ax^2 + C) = 3x^2 + 4$$

$\Rightarrow 2Ax^2 - 8Ax^2 + 4Ax^2 + 4C = 3x^2 + 4$

i.e., $-2Ax^2 + 4C = 3x^2 + 4$

$\Rightarrow -2A = 3 \Rightarrow A = -\frac{3}{2}$

Also: $4C = 4 \Rightarrow C = 1$

$\Rightarrow y_p = Ax^2 + C = -\frac{3}{2}x^2 + 1$

The solution to the original equation is: $y = y_p + y_c$

The solution to the original equation is: $y_g = -\frac{3}{2}x^2 + 1 + c_1x + c_2x^4$

3. Solve the Differential Equation: $y'' - 3y' + 2y = xe^{2x}$

First, we must find the complementary solution, y_c

We consider the complementary equation $y'' - 3y' + 2y = 0$

Using differential operator notation, this becomes:

$D^2y - 3Dy + 2y = 0$

$\Rightarrow \underbrace{\left(D^2 - 3D + 2\right)}_{\phi(D)} y = 0$

$\Rightarrow \phi(D) = D^2 - 3D + 2$

We find the roots of the auxilliary equation: $\phi(m) = 0$

$\Rightarrow \underbrace{m^2 - 3m + 2}_{\phi(m)} = 0$

$\Rightarrow (m-1)(m-2) = 0$

$\Rightarrow m_1 = 1$ and $m_2 = 2$

Our Complementary Solution is $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y_c = c_1 e^{1x} + c_2 e^{2x}$

i.e., $y_c = c_1 e^{x} + c_2 e^{2x}$

Next, we find a particular solution y_p to the equation:

$y'' - 3y' + 2y = xe^{2x}$

Since the right hand side is NOT a polynomial, or a linear combination of sines, cosines, and exponentials, we can NOT use the Method of Undetermined Coefficients.

We have to use the Method of Variation of Parameters

To do this, we form our general solution by taking our complementary solution $y_c = c_1 e^{1x} + c_2 e^{2x}$ and replacing the arbitrary constants with functions of x.

$y = A(x)\, e^{x} + B(x)\, e^{2x}$

Since we have two arbitrary functions of x in our solution, we are allowed to impose two restrictions on this pair of functions.

The first restriction that we impose is that the pair of functions $(A(x), B(x))$ are such that $y = A(x)e^x + B(x)e^{2x}$ is a solution of the equation: $y = A(x)e^x + B(x)e^{2x}$.

This is ALWAYS the first restriction that we impose on the pair of functions $(A(x), B(x))$.

Next, we compute the derivative of y

$y = A(x)e^x + B(x)e^{2x}$

$$y' = \underbrace{A(x)e^x + e^x A'(x)}_{\text{Product Rule}} + \underbrace{B(x)2e^{2x} + e^{2x}B'(x)}_{\text{Product Rule}}$$

At this point, we impose the second of the two restrictions that we are allowed to impose on the pair of functions $(A(x), B(x))$.

To rid $f'(x)$ of the terms that contain derivatives of $A(x)$ and $B(x)$, we impose the restriction that **the sum of the terms that contain $A'(x)$ and $B'(x)$ equals zero.**

$\Rightarrow e^x A'(x) + e^{2x}B'(x) = 0$

In this case, this step yields:

$y' = A(x)e^x + B(x)2e^{2x}$

Remark: The second restriction that we imposed (Setting the sum of the terms that contain $A'(x)$ and $B'(x)$ equal to zero) is **almost always** the second restriction that we impose.

Finally, we compute y''

$$y'' = \underbrace{A(x)e^x + e^x A'(x)}_{\text{Product Rule}} + \underbrace{B(x)4e^{2x} + 2e^{2x}B'(x)}_{\text{Product Rule}}$$

Next, we plug the expressions for y, y', and y'' into the equation: $y'' - 3y' + 2y = xe^{2x}$

$$\begin{array}{rcccccccc} & y'' & A(x)e^x & + & e^x A'(x) & + & 4B(x)e^{2x} & + & 2e^{2x}B'(x) & & \\ - & 3y' & -3A(x)e^x & & & - & 6B(x)e^{2x} & & & & \\ + & 2y & 2A(x)e^x & & & + & 2B(x)e^{2x} & & & & \\ \hline y''-3y'+2y & = & & & e^x A'(x) & & & + & 2e^{2x}B'(x) & = & xe^{2x} \end{array}$$

i.e., $e^x A'(x) + 2e^{2x}B'(x) = xe^{2x}$

Our immediate goal is to use this equation to solve for functions $A(x)$ and $B(x)$.

But we have TWO unknown functions, $A(x)$ and $B(x)$, and only ONE equation.

How can we solve for these functions?

The answer is that we can still apply the second restriction that we imposed on $(A(x), B(x))$.

We can use the second restriction to eliminate one of the functions from the equation

Observe:

$$\begin{array}{lrcrcr} & & e^x A'(x) & + & 2e^{2x} B'(x) & = & xe^{2x} \\ \text{Restriction:} \rightarrow & - & (e^x A'(x) & + & e^{2x} B'(x)) & = & 0 \\ \hline & & & & e^{2x} B'(x) & = & xe^{2x} \end{array}$$

This yields:

$\Rightarrow e^{2x} B'(x) = xe^{2x}$

$\Rightarrow B'(x) = x$

$\Rightarrow B(x) = \int B'(x)\,dx = \int x dx = \frac{1}{2}x^2 + C_2$

i.e., $B(x) = \frac{1}{2}x^2 + C_2$

Now how do we solve for $A(x)$?

Recall: $B'(x) = x$

Perhaps the best way to solve for $A(x)$ is to take $B'(x) = x$ and plug this into the equation:

$e^x A'(x) + e^{2x} B'(x) = 0$ (our restriction)

$\Rightarrow e^x A'(x) + e^{2x} \cdot x = 0$

$\Rightarrow A'(x) + x\frac{e^{2x}}{e^x} = 0$

$\Rightarrow A'(x) + xe^x = 0$

$\Rightarrow A'(x) = -xe^x$

$\Rightarrow A(x) = \int A'(x)\,dx = -\int xe^x dx$

i.e., $A(x) = -\int xe^x dx$

We have to use "integration by parts" to do this

$-\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = -\left[uv - \int vdu\right] = -\left[xe^x - \int e^x dx\right] = -\left[xe^x - e^x\right] + C_1 = -xe^x + e^x + C_1$

	$u = x$		$dv = e^x dx$
$\Rightarrow$	$\frac{du}{dx} = 1$	$\Rightarrow$	$\int dv = \int e^x dx$
$\Rightarrow$	$du = dx$	$\Rightarrow$	$v = e^x$

i.e., $A(x) = -xe^x + e^x + C_1$

So we have established that $A(x) = -xe^x + e^x + C_1$ and that $B(x) = \frac{1}{2}x^2 + C_2$

Plugging these into $y = A(x)e^x + B(x)e^{2x}$, we have:

$$\boxed{y = (-xe^x + e^x + C_1)e^x + \left(\tfrac{1}{2}x^2 + C_2\right)e^{2x}}$$

This can be rewritten as:

$$\boxed{y = \left(\tfrac{1}{2}x^2 - x\right)e^{2x} + C_1e^x + C_3e^{2x}}$$