MTH 1126 - Test #3 - Solutions

SPRING 2006

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Instructions. Show CLEARLY how you arrive at your answers

1. $\int \frac{1}{\sqrt{9x^2+4}}dx =$

$$\begin{array}{lll} & a^2 & = 4 \\ \Rightarrow & a & = 2 \\ \Rightarrow & 9x^2 & = a^2\tan^2(\theta) = 4\tan(\theta) \\ \Rightarrow & 3x & = 2\tan(\theta) \\ \Rightarrow & x & = \frac{2}{3}\tan(\theta) \\ \Rightarrow & \frac{dx}{d\theta} & = \frac{2}{3}\sec^2(\theta) \\ \Rightarrow & dx & = \frac{2}{3}\sec^2(\theta)\,d\theta \end{array}$$

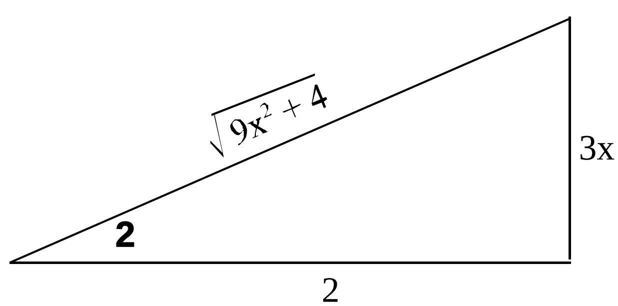
$$\int \frac{1}{\sqrt{9x^2+4}}dx = \int \underbrace{\frac{1}{\sqrt{4+4\tan^2(\theta)}}}_{\frac{1}{\sqrt{9x^2+4}}} \underbrace{\frac{2}{3}\sec^2(\theta)\ d\theta}_{dx} = \int \frac{1}{\sqrt{4\sec^2(\theta)}}\frac{2}{3}\sec^2(\theta)\ d\theta$$

$$= \frac{2}{3}\int \frac{1}{2\sec(\theta)}\sec^2(\theta)\ d\theta = \frac{1}{3}\int \sec(\theta)\ d\theta = \frac{1}{3}\ln|\sec(\theta)+\tan(\theta)| + C$$

Now, convert back to x.

Recall:

$$\begin{array}{lll} & x & = \frac{2}{3}\tan(\theta) \\ \Rightarrow & \frac{3x}{2} & = \tan(\theta) \end{array}$$



Recall: $\int \frac{1}{\sqrt{9x^2+4}}dx = \frac{1}{3}\ln|\sec(\theta)+\tan(\theta)| + C = \frac{1}{3}\ln\left|\frac{\sqrt{9x^2+4}}{2} + \frac{3x}{2}\right| + C$

$= \frac{1}{3}\ln\left|\frac{\sqrt{9x^2+4}+3x}{2}\right| + C$

i.e., $\int \frac{1}{\sqrt{9x^2+4}}dx = \frac{1}{3}\ln\left|\frac{\sqrt{9x^2+4}+3x}{2}\right| + C$

2. $\int_2^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_2^b x^{-2}dx = \lim_{b\to\infty}\left[\frac{x^{-1}}{-1}\right]_2^b = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_2^b$

$= \lim_{b\to\infty}\left[-\frac{1}{b} - \left(-\frac{1}{2}\right)\right] = \frac{1}{2}$

The integral converges and is equal to $\frac{1}{2}$ (i.e., $\int_2^\infty \frac{1}{x^2}dx = \frac{1}{2}$)

3. $\lim_{x\to 0}\frac{\cos(x)-1}{e^x-e^{-x}} \sim \frac{0}{0}$ (use L'Hôpital's Rule)

$\lim_{x\to 0}\frac{\cos(x)-1}{e^x-e^{-x}} = \lim_{x\to 0}\frac{-\sin(x)}{e^x+e^{-x}} = \frac{0}{2} = 0$

i.e., $\lim_{x\to 0}\frac{\cos(x)-1}{e^x-e^{-x}} = 0$

For Problems 4-7, determine whether the given series converges or diverges. If the series converges, determine its sum.

4. $\sum_{n=1}^\infty \frac{1}{n} = \frac{1}{n} + \frac{1}{2} + \frac{1}{3} + \ldots$

This is the *Harmonic Series.* It diverges.

5. $\sum_{n=1}^\infty \frac{5}{3^n} = \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \ldots$

Note that each term after the first is obtained by multiplying its predecessor by $\frac{1}{3}$.

Thus, the series is *geometric* with ratio, $r = \frac{1}{3}$

Since $|r| < 1,$ the series converges and has sum $\frac{1^{st}\text{ term}}{1-r} = \frac{\left(\frac{5}{3}\right)}{1-\frac{1}{3}} = \frac{5}{3}\cdot\frac{3}{2} = \frac{5}{2}$

i.e., the series converges geometrically, and $\sum_{n=1}^\infty \frac{5}{3^n} = \frac{5}{2}$

6. $\sum_{n=1}^\infty \frac{n^2}{2n^2+1} = \frac{1}{3} + \frac{4}{9} + \frac{9}{19} + \ldots$

Observe: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^2}{2n^2+1} = \lim_{n\to\infty}\frac{n^2}{2n^2} = \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2}$

Since $\lim_{n\to\infty} a_n \neq 0,$ the series diverges

7. $\sum_{n=1}^{\infty} \frac{2}{n^2+2n} = \frac{2}{3} + \frac{2}{8} + \frac{2}{15} + \dots$

Observe: $\frac{2}{n^2+2n} = \frac{2}{n(n+2)} = \frac{C_1}{n} + \frac{C_2}{n+2}$

i.e., $\frac{2}{n(n+2)} = \frac{C_1}{n} + \frac{C_2}{n+2}$

$\Rightarrow 2 = \frac{C_1}{n} n\,(n+2) + \frac{C_2}{n+2} n\,(n+2) = C_1\,(n+2) + C_2 n$

i.e., $2 = C_1\,(n+2) + C_2 n$

$\boxed{n = 0}$

$\Rightarrow 2 = 2C_1$

$\Rightarrow C_1 = 1$

$\boxed{n = -2}$

$\Rightarrow 2 = -2C_2$

$\Rightarrow C_2 = -1$

Thus, $\frac{2}{n^2+2n} = \frac{2}{n(n+2)} = \frac{C_1}{n} + \frac{C_2}{n+2} = \frac{1}{n} - \frac{1}{n+2}$

And we have: $\frac{2}{n^2+2n} = \frac{1}{n} - \frac{1}{n+2}$

$\sum_{n=1}^{\infty} \frac{2}{n^2+2n} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{2}{n^2+2n} = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

$= \lim_{N\to\infty} \Big[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots$

$\dots + \left(\frac{1}{N-2} - \frac{1}{N}\right) + \left(\frac{1}{N-1} - \frac{1}{N+1}\right) + \left(\frac{1}{N} - \frac{1}{N+2}\right)\Big]$

$= \lim_{N\to\infty} \left[1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right] = \frac{3}{2}$

$\boxed{\text{Since this limit is finite, the series converges and is equal to } \frac{3}{2} \text{ (i.e., } \sum_{n=1}^{\infty} \frac{2}{n^2+2n} = \frac{3}{2}\text{)}}$