MTH 1125 Test #3 - Solutions

Summer 2022

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Instructions. Show CLEARLY how you arrive at your answers.

1. $f(x) = 2x^3 - 3x^2 - 12x + 5$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

   i. Compute $f'(x)$ and find critical numbers

   $f'(x) = 6x^2 - 6x - 12$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 6x^2 - 6x - 12 = 0$

   $\Rightarrow 6x^2 - 6x - 12 = 0$

   $\Rightarrow x^2 - x - 2 = 0$

   $\Rightarrow (x+1)(x-2) = 0$

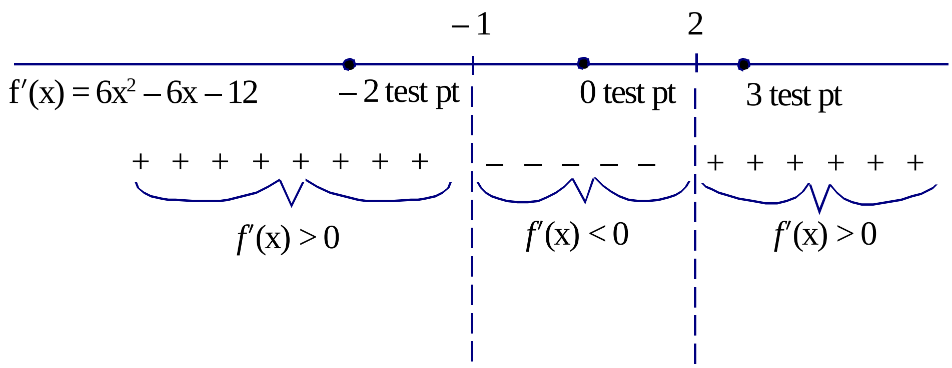
   $\Rightarrow x = -1;\ x = 2$ critical numbers

   b. "Type b" $(f'(c)$ undefined$)$

   There are none.

   ii. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

   iii. From each interval select a "test point" to plug into $f'(x)$



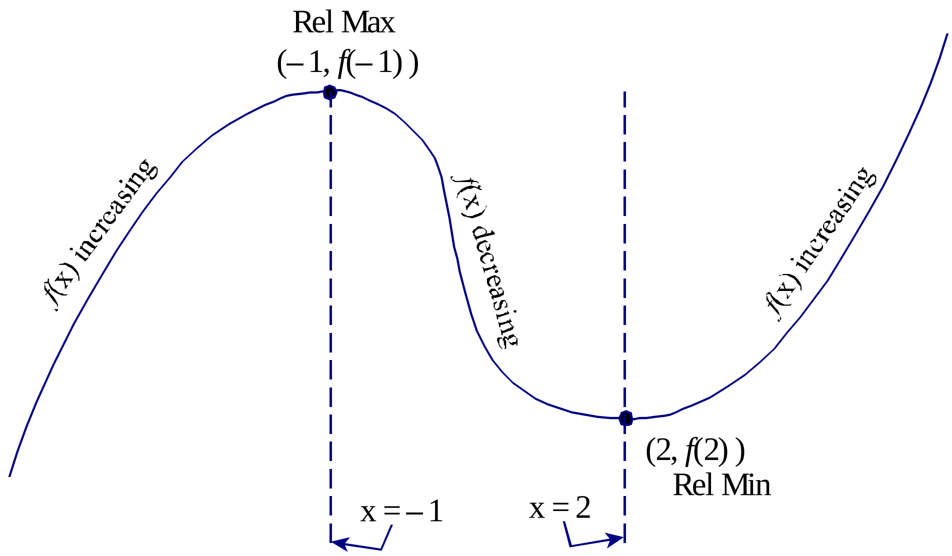
   $f(x)$ is **increasing** on the intervals $(-\infty, -1)$ and $(2, \infty)$

   (Because $f'(x)$ is positive on these intervals)

   $f(x)$ is **decreasing** on the interval $(-1, 2)$

   (Because $f'(x)$ is negative on this interval)

iv. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



$f(x)$ is **increasing** on the intervals $(-\infty, -1)$ and $(2, \infty)$

$f(x)$ is **decreasing** on the interval $(-1, 2)$

$(-1, f(-1)) = (-1, 12)$ Relative Max

$(2, f(2)) = (2, -15)$ Relative Min

2. $f(x) = x^4 + 2x^3 - 12x^2 + 6x + 3$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection.

   i. Compute $f''(x)$ and find possible points of inflection

      $f'(x) = 4x^3 + 6x^2 - 24x + 6$

      $f''(x) = 12x^2 + 12x - 24$

      a. "Type a" $(f''(c) = 0)$

         Set $f''(x) = 12x^2 + 12x - 24 = 0$

         $\Rightarrow 12x^2 + 12x - 24 = 0$

         $\Rightarrow x^2 + x - 2 = 0$

         $\Rightarrow (x+2)(x-1) = 0$

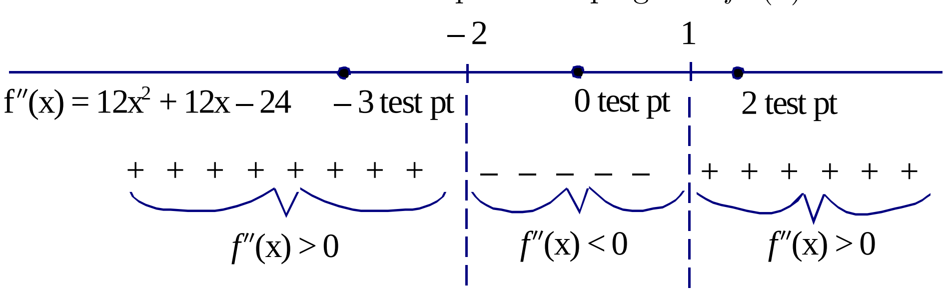
         $\Rightarrow x = -2$; and $x = 1$ possible points of inflection

      b. "Type b" $(f''(c)$ undefined$)$

         There are none.

   ii. Draw a sign graph of $f''(x)$, using the possible points of inflection to partition the x-axis

   iii. From each interval select a "test point" to plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-\infty, -2)$ and $(1, \infty)$

(Because $f''(x) > 0$ on these intervals)

$f(x)$ is **concave down** on the interval $(-2, 1)$

(Because $f''(x) < 0$ on this interval)

Since $f(x)$ changes concavity at $x = -2$ and $x = 1$, the points:

$(-2, f(-2)) = (-2, -57)$

and

$(1, f(1)) = (1, 0)$ *are* points of inflection.

3. $f(x) = 2x^3 - 9x^2 + 3$ on the interval $[-2, 2]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

Since $f(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[-2, 2]$, we can use the Absolute Max/Min Value Test

i. Compute $f'(x)$ and find the critical numbers

$f'(x) = 6x^2 - 18x$

a. "Type a" ($f'(c) = 0$)

Set $f'(x) = 6x^2 - 18x = 0$

$\Rightarrow 6x^2 - 18x = 0$

$\Rightarrow 6x(x - 3) = 0$

$\Rightarrow x = 0;\ x = 3$ "type a" crit. numbers

Since $3 \notin [-2, 2]$, we discard $x = 3$ as a critical number.

b. "Type b" ($f'(c)$ undefined)

There are none.

ii. Plug critical numbers and endpoints into the original function.

$f(-2) = 2(-2)^3 - 9(-2)^2 + 3 = -49 \leftarrow$ Abs Min Value

$f(0) = 2(0)^3 - 9(0)^2 + 3 = 3 \leftarrow$ Abs Max Value

$f(2) = 2(2)^3 - 9(2)^2 + 3 = -17$

Abs Max Value $= 3$
(attained at $x = 0$)

Abs Min Value $= -49$
(attained at $x = -2$)

4. $f(x) = x^{\frac{12}{5}} - 6x^{\frac{2}{5}} + 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

   i. Compute $f'(x)$ and find critical numbers

   $f'(x) = \frac{12}{5}x^{\frac{7}{5}} - \frac{12}{5}x^{-\frac{3}{5}} = \frac{12x^{\frac{7}{5}}}{5} - \frac{12}{5x^{\frac{3}{5}}} = \frac{12x^{\frac{7}{5}}}{5}\frac{x^{\frac{3}{5}}}{x^{\frac{3}{5}}} - \frac{12}{5x^{\frac{3}{5}}} = \frac{12x^{\frac{10}{5}}}{5x^{\frac{3}{5}}} - \frac{12}{5x^{\frac{3}{5}}} = \frac{12x^2-12}{5x^{\frac{3}{5}}}$

   i.e., $f'(x) = \frac{12x^2-12}{5x^{\frac{3}{5}}}$

   a. "Type a" $(f'(c) = 0)$

      Set $f'(x) = \frac{12x^2-12}{5x^{\frac{3}{5}}} = 0$

      $\Rightarrow 12x^2 - 12 = 0$

      $\Rightarrow x^2 - 1 = 0$

      $\Rightarrow (x+1)(x-1) = 0$

      $\Rightarrow x = -1;\ x = 1$ ("type a" critical numbers)

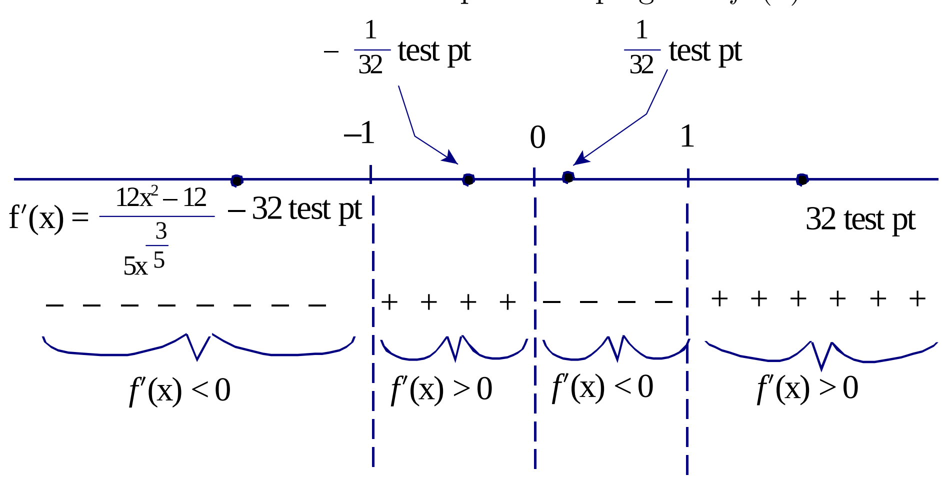
   b. "Type b" ($f'(c)$ undefined)

      $f'(x)$ is undefined when $5x^{\frac{3}{5}} = 0$

      $\Rightarrow x = 0$ ("type a" critical numbers)

   ii. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

   iii. From each interval select a "test point" to plug into $f'(x)$



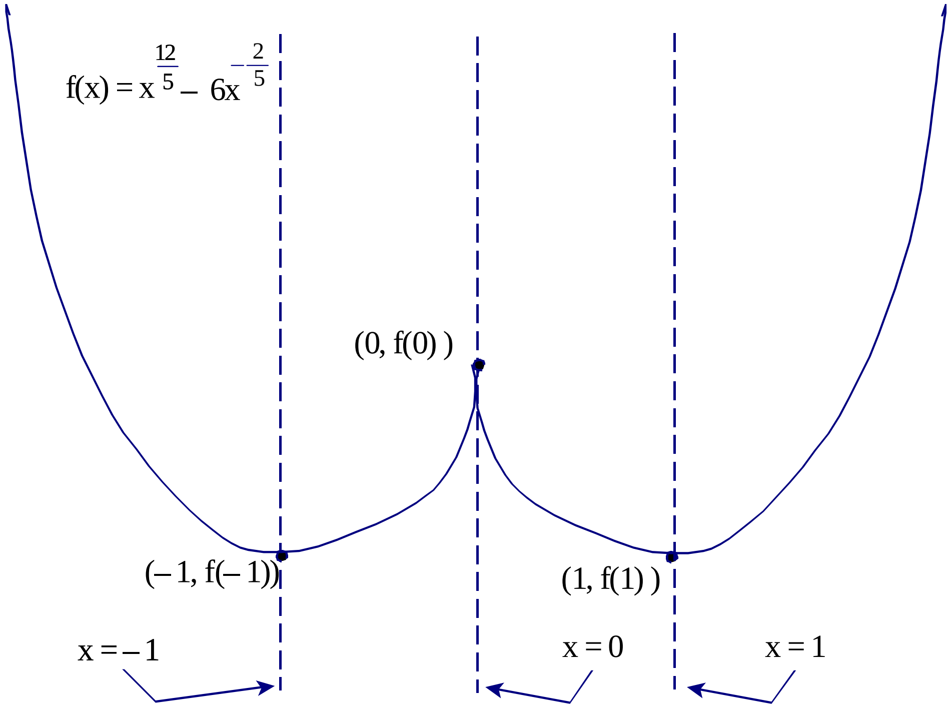
$f(x)$ is **increasing** on the intervals $(-1, 0)$ and $(1, \infty)$

(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the intervals $(-\infty, -1)$ and $(0, 1)$

(Because $f'(x)$ is negative on these intervals)

iv. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



$f(x)$ is **increasing** on the intervals $(-1, 0)$ and $(1, \infty)$

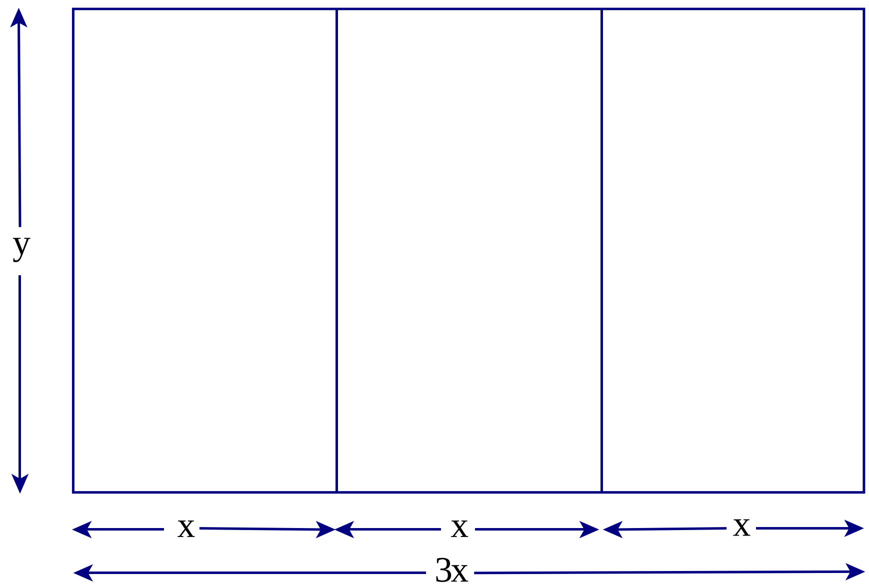
$f(x)$ is **decreasing** on the intervals $(-\infty, -1)$ and $(0, 1)$ $x^{\frac{12}{5}} - 6x^{\frac{2}{5}} + 2$

$(-1, f(-1)) = (-1, -3)$ Relative Min

$(0, f(0)) = (0, 2)$ Relative Max

$(1, f(1)) = (1, -3)$ Relative Min

5. A rancher has 200 yards of fencing to enclose three adjacent rectangular corrals, as shown below. What overall dimensions should be used so that the enclosed area will be as large as possible?



i. Determine the quantity to be maximized/minimized - give it a name,

   Maximize the overall area of the pen, $A = 3xy$

ii. Express A as a function of *one* variable.

   (Refer to a restriction stated in the problem to do this)

   Restriction: Farmer Joe will use exactly 200 yards of fencing.

   Since the fencing consists of six segments of length y and four segments of length x, we have:

   $6x + 4y = 200$ yds

   $\Rightarrow 4y = 200 \text{ yds } - 6x$

   $\Rightarrow y = 50 \text{ yds } - \frac{3}{2}x$

   Substituting this into the equation $A = 3xy$, we have:

   $A = 3x\left(50 \text{ yds} - \frac{3}{2}x\right) = 150 \text{ yds } x - \frac{9}{2}x^2$

   i.e., $A(x) = 150 \text{ yds } x - \frac{9}{2}x^2$

iii. Determine the restrictions on y

   $0 \text{ yds} \leq x \leq \frac{100}{3} \text{ yds}$

iv. Maximize/minimize, using the techniques of calculus.

Observe: $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $\left[0 \text{ yds}\ ,\ \frac{100}{3} \text{ yds}\right]$.

Therefore, we can use the Absolute Max/Min Value Test

1. Compute $A'(x)$ and find the critical numbers

   $A'(x) = 150 \text{ yds } - 9x$

   a. "Type a" $(A'(c) = 0)$

      $A'(x) = 150 \text{ yds } - 9x = 0$

      $\Rightarrow 150 \text{ yds } - 9x = 0$

      $\Rightarrow 9x = 150 \text{ yds}$

      $\Rightarrow x = \frac{50}{3}$ yds - critical number

   b. "Type b" $(A'(c)$ is undefined$)$

      There are none.

2. Plug the critical numbers and endpoints into the *original function.*

   $A(0 \text{ yds}) = 150\,(0 \text{ yds}) - \frac{9}{2}(0 \text{ yds})^2 = 0 \text{ yds}^2$

   $A\left(\frac{50}{3} \text{ yds}\right) = 150\left(\frac{50}{3} \text{ yds}\right) - \frac{9}{2}\left(\frac{50}{3} \text{ yds}\right)^2 = 1250 \text{ yds}^2 \leftarrow$ Abs Max Value

   $A\left(\frac{100}{3} \text{ yds}\right) = 150\left(\frac{100}{3} \text{ yds}\right) - \frac{9}{2}\left(\frac{100}{3} \text{ yds}\right)^2 = 0 \text{ yds}^2$

5. Make sure that we've solved the original question (problem).

"What should the overall dimensions ... be"

We have the Abs Max Area when $x = \frac{50}{3}$ yds

Length $= 3\text{x} = 3\left(\frac{50}{3} \text{ yds}\right) = 50$ yds

Width $= y = 50 \text{ yds } - \frac{3}{2}x = 50 \text{ yds} - \frac{3}{2}\left(\frac{50}{3} \text{ yds}\right) = 25$ yds

Length = 50 yds
Width = 25 yds

(Alternative Solution on the next page)

i. Determine the quantity to be maximized/minimized - give it a name,

Maximize the overall area of the pen, $A = 3xy$

ii. Express A as a function of *one* variable.

(Refer to a restriction stated in the problem to do this)

Restriction: Farmer Joe will use exactly 200 yards of fencing.

Since the fencing consists of six segments of length x and four segments of length y, we have:

$6x + 4y = 200$ yds

$\Rightarrow 6x = 200$ yds $-4y$

$\Rightarrow 3x = 100$ yds $-2y$

Substituting this into the equation $A = 3xy$, we have:

$A = (100 \text{ yds } - 2y)\, y = 100 \text{ yds } y - 2y^2$

i.e., $A(y) = 100 \text{ yds } y - 2y^2$

iii. Determine the restrictions on y

$0 \text{ yds} \leq y \leq 50 \text{ yds}$

iv. Maximize/minimize, using the techniques of calculus.

Observe: $A(y)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0 \text{ yds}, 50 \text{ yds}]$.

Therefore, we can use the Absolute Max/Min Value Test

1. Compute $A'(y)$ and find the critical numbers

   $A'(y) = 100$ yds $-4y$

   a. "Type a" $(A'(c) = 0)$

      $A'(y) = 100$ yds $-4y = 0$

      $\Rightarrow 100$ yds $-4y = 0$

      $\Rightarrow 4y = 100$ yds

      $\Rightarrow y = 25$ yds - critical number

   b. "Type b" ($A'(c)$ is undefined)

      There are none.

2. Plug the critical numbers and endpoints into the *original function.*

   $A\,(0 \text{ yds}) = 100\,(0 \text{ yds}) - 2\,(0 \text{ yds})^2 = 0 \text{ yds}^2$

   $A\,(25 \text{ yds}) = 100\,(25 \text{ yds}) - 2\,(25 \text{ yds})^2 = 1250 \text{ yds}^2 \leftarrow$ Abs Max Value

   $A\,(50 \text{ yds}) = 100\,(50 \text{ yds}) - 2\,(50 \text{ yds})^2 = 0 \text{ yds}^2$

5. Make sure that we've solved the original question (problem).

   "What should the overall dimensions ... be"

   We have the Abs Max Area when $y = 25$ yds

   Width $= y = 25$ yds

   Length $= 3\text{x} = 100$ yds $-2y = 50$ yds

   Length $= 50$ yds

   Width $= 25$ yds