MTH 1126 Test #3 - Solutions

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Show CLEARLY how you arrive at your answers.

1. $\int x\cos(x)\,dx =$

   Use Integration by Parts:

   $$\int x\cos(x)\ dx = \int \underbrace{x}_{u}\underbrace{\cos(x)\ dx}_{dv} = \int udv = uv - \int vdu = \underbrace{x}_{u}\cdot\underbrace{(\sin(x))}_{v} - \int\underbrace{(\sin(x))}_{v}\cdot\underbrace{dx}_{du}$$

   $$= x\sin(x) - [-\cos(x)] = x\sin(x) + \cos(x) + C$$

u	$=$	x	dv	$=$	$\cos(x)\,dx$
$\frac{du}{dx}$	$=$	1	$\int dv$	$=$	$\int \cos(x)\,dx$
du	$=$	dx	v	$=$	$\sin(x)$

   i.e. $\int x\cos(x)\ dx = x\sin(x) + \cos(x) + C$

2. $\lim_{x\to 0}\frac{e^x - e^{-x}}{x} = \frac{1-1}{0} \sim \frac{0}{0}$ So use L'Hôpital's Rule.

   $\lim_{x\to 0}\frac{e^x - e^{-x}}{x} = \lim_{x\to 0}\frac{e^x + e^{-x}}{1} = \frac{2}{1} = 2$

   i.e., $\lim_{x\to 0}\frac{e^x - e^{-x}}{x} = 2$

3. $\int \frac{5x-4}{(x^2-x-2)}dx =$

   Express as $\int \frac{5x-4}{(x+1)(x-2)}dx$

   Observe:

   $\frac{5x-4}{(x+1)(x-2)} = \frac{C_1}{x+1} + \frac{C_2}{x-2}$

   $\Rightarrow 5x - 4 = C_1(x-2) + C_2(x+1)$

   Solve for the constants, by plugging in "strategic values" of x.

   $x = 2$ We have: $6 = 3C_2 \Rightarrow \mathbf{C_2 = 2}$

   $x = -1$ We have: $-9 = -3C_1 \Rightarrow \mathbf{C_1 = 3}$.

   Therefore:

   $$\int \frac{5x-4}{(x+1)(x-2)}dx = \int\left(\frac{3}{x+1} + \frac{2}{x-2}\right)dx = \int\frac{3}{x+1}dx + \int\frac{2}{x-2}dx = 3\ln|x+1| + 2\ln|x-2| + C$$

   i.e., $\int \frac{5x-4}{(x^2-x-2)}dx = 3\ln|x+1| + 2\ln|x-2| + C$

4. $\int \sin^3(x)\cos^4(x)\,dx =$

(sine to an odd power) Pull out a factor of $\sin(x)$ to serve as the "future du", and convert the rest of the sines to cosines using the identity $\sin^2(x) = 1 - \cos^2(x)$.

$$\int \sin^3(x)\cos^4(x)\,dx = \int \sin^2(x)\cos^4(x)\underbrace{\sin(x)\ dx}_{\text{future } du} == \int \left(1-\cos^2(x)\right)\cos^4(x)\sin(x)\,dx.$$

$$\boxed{\begin{aligned} \text{Let } u &= \cos(x) \\ \Rightarrow du &= -\sin(x)\,dx \\ \Rightarrow -du &= \sin(x)\,dx \end{aligned}}$$

Continuing, we have:

$$\int \underbrace{\left(1-\cos^2(x)\right)}_{(1-u^2)}\underbrace{\cos^4(x)}_{u^4}\underbrace{\sin(x)\,dx}_{-du} = \int \left(1-u^2\right)u^4(-du) = \int \left(u^2-1\right)u^4(du) = \int \left(u^6-u^4\right)(du)$$

$$= \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C$$

$$= \tfrac{1}{7}\cos^7(x) - \tfrac{1}{5}\cos^5(x) + C$$

$$\boxed{\text{i.e., } \int \sin^3(x)\cos^4(x)\,dx = \tfrac{1}{7}\cos^7(x) - \tfrac{1}{5}\cos^5(x) + C}$$

5. $\int \frac{1}{x^2\sqrt{9-16x^2}}dx =$

Use trig substitution to get rid of the radical. We want to replace $\sqrt{9-16x^2}$ with something of the form $a^2 - a^2\sin^2(\theta)$.

Thus, we have:

$$a^2 = 9$$

$$\Rightarrow a = 3$$

Also:

$$16x^2 = a^2\sin^2(\theta) = 9\sin^2(\theta)$$

$$\Rightarrow 16x^2 = 9\sin^2(\theta)$$

$$\Rightarrow 4x = 3\sin(\theta)$$

$$\Rightarrow x = \frac{3}{4}\sin(\theta)$$

$$\Rightarrow 1 = \frac{3}{4}\cos(\theta)\frac{d\theta}{dx}$$

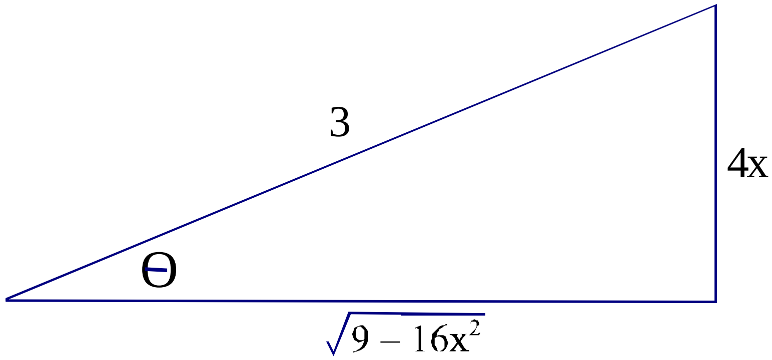
$$\Rightarrow dx = \frac{3}{4}\cos(\theta)\,d\theta$$

Continuing with our integration, we have:

$$\int \frac{1}{x^2\sqrt{9-16x^2}}dx = \int \frac{1}{\left(\frac{3}{4}\sin(\theta)\right)^2\sqrt{9-9\sin^2(\theta)}}\frac{3}{4}\cos(\theta)\,d\theta = \int \frac{1}{\left(\frac{9}{16}\sin^2(\theta)\right)\sqrt{9\cos^2(\theta)}}\frac{3}{4}\cos(\theta)\,d\theta$$

$$= \int \frac{16}{9}\frac{1}{\sin^2(\theta)(3\cos(\theta))}\frac{3}{4}\cos(\theta)\,d\theta = \int \frac{4}{9}\frac{1}{\sin^2(\theta)}d\theta = \frac{4}{9}\int \csc^2(\theta)\,d\theta = -\frac{4}{9}\cot(\theta) + C$$

Observe: $x = \frac{3}{4}\sin(\theta) \Rightarrow \frac{4x}{3} = \sin(\theta) = \frac{\text{opp}}{\text{hyp}}$. This yields the triangle below:



Continuing, we have:

$$\int \frac{1}{x^2\sqrt{9-16x^2}}dx = -\frac{4}{9}\cot(\theta) + C = -\frac{4}{9}\frac{\sqrt{9-16x^2}}{4x} + C = -\frac{\sqrt{9-16x^2}}{9x} + C$$

$$\boxed{\text{i.e., } \int \frac{1}{x^2\sqrt{9-16x^2}}dx = -\frac{4}{9}\frac{\sqrt{9-16x^2}}{4x} + C = -\frac{\sqrt{9-16x^2}}{9x} + C}$$

6. $\lim_{x\to\infty} \frac{e^x}{x^2} \sim \frac{\infty}{\infty}$ So we can use L'Hôpital's Rule.

$\lim_{x\to\infty} \frac{e^x}{x^2} = \underbrace{\lim_{x\to\infty} \frac{e^x}{2x}}_{\sim\frac{\infty}{\infty}} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$

i.e., $\lim_{x\to\infty} \frac{e^x}{x^2} = \infty$

7. $\int \cos^2(x)\, dx =$

(This is a special case of sine and cosine raised to even powers).
First, express in terms of $\sin^2(x)$ and $\cos^2(x)$.

(Done.)

Next, Reduce the powers of sine and cosine by using the double angle formulas:

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \quad \text{and} \quad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

Continuing, we have:

$\int \cos^2(x)\, dx = \int \frac{1+\cos(2x)}{2} dx = \frac{1}{2} \int (1+\cos(2x))\, dx$

$$= \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$

i.e., $\int \cos^2(x)\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$