MTH 1126 - Test #4 - Version 1 - Solutions

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Show CLEARLY how you arrive at your answers.

In Exercises 1-2, Determine convergence/divergence. If the integral converges, find its value.

1. $\int_6^\infty \frac{1}{(x-2)^{\frac{3}{2}}}dx =$

$\int_6^\infty \frac{1}{(x-2)^{\frac{3}{2}}}dx = \lim_{b\to\infty} \int_6^b \frac{1}{(x-2)^{\frac{3}{2}}}dx = \lim_{b\to\infty} \int_6^b (x-2)^{-\frac{3}{2}}\,dx = \lim_{b\to\infty}\left[\frac{(x-2)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)}\right]_6^b$

$= \lim_{b\to\infty}\left[-2(x-2)^{-\frac{1}{2}}\right]_6^b = \lim_{b\to\infty}\left[-\frac{2}{(x-2)^{\frac{1}{2}}}\right]_6^b$

$= \lim_{b\to\infty}\left[-\frac{2}{(b-2)^{\frac{1}{2}}} - \left(-\frac{2}{(6-2)^{\frac{1}{2}}}\right)\right] = \lim_{b\to\infty}\left[-\frac{2}{(b-2)^{\frac{1}{2}}} + \frac{2}{(6-2)^{\frac{1}{2}}}\right]$

$= \lim_{b\to\infty}\left[-\frac{2}{(b-2)^{\frac{1}{2}}} + \frac{2}{2}\right] = [0+1] = 1$

i.e., $\int_6^\infty \frac{1}{(x-2)^{\frac{3}{2}}}dx = 1$ (Integral **Converges)**

2. $\int_2^6 \frac{1}{(x-2)^{\frac{1}{2}}}dx =$

Because $\frac{1}{(x-2)^{\frac{1}{2}}}$ is discontinuous at $x = 2$, this is an improper integral.

$\int_2^6 \frac{1}{(x-2)^{\frac{1}{2}}}dx = \lim_{a\to2^+} \int_a^6 \frac{1}{(x-2)^{\frac{1}{2}}}dx = \lim_{a\to2^+} \int_a^6 (x-2)^{-\frac{1}{2}}\,dx = \lim_{a\to2^+}\left[\frac{(x-2)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right]_a^6$

$= \lim_{a\to2^+}\left[2(x-2)^{\frac{1}{2}}\right]_a^6 = \lim_{a\to2^+}\left[2(6-2)^{\frac{1}{2}} - 2(a-2)^{\frac{1}{2}}\right]$

$= \left[2(2) - 2(0)^{\frac{1}{2}}\right] = 4$

i.e. $\int_2^6 \frac{1}{(x-2)^{\frac{1}{2}}}dx = 4$ (Integral **Converges)**

3. Determine convergence/divergence of the sequence whose n^{th} term is given by: $a_n = \cos\left(\frac{n\pi}{2}\right)$. (i.e., Determine convergence/divergence of the sequence $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}_{n=1}^{\infty} = \{0, 1, 0, -1, \ldots\}$.)

   Observe: $\lim_{n\to\infty} \cos\left(\frac{n\pi}{2}\right)$ Does Not Exist.

   The logic here is that in order for the sequence to converge to some real number L, the terms of the sequence $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}_{n=1}^{\infty}$ have to get arbitrarily close to L as n gets large. However, consecutive terms of the sequence will always be 1 unit apart, which means that if one term of the sequence is extremely close to L, its successor and predecessor will have to be almost 1 unit away from L.

   Therefore, $\lim_{n\to\infty} \cos\left(\frac{n\pi}{2}\right)$ Does Not Exist, and consequently, the series Diverges.

The Sequence **Diverges.** ($\lim_{n\to\infty} \cos\left(\frac{n\pi}{2}\right)$ Does Not Exist)

4. Determine convergence/divergence of the given series. (Justify your answer!) **If the series converges, determine its sum.**

   $$\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6} =$$

   If the series converges, determine its sum. In general, there are only two types of convergent series whose sums we can compute: Geometric and "Telescoping Sum."

   The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6}$ is definitely NOT Geometric.

   Maybe it can be written as a "Telescoping Sum."

   So let's see if we can express $a_n = \frac{1}{n^2+5n+6}$ as the difference of two terms.

   $\frac{1}{n^2+5n+6} = \frac{1}{(n+2)(n+3)} = \frac{C_1}{n+2} + \frac{C_2}{n+3}$

   i.e., $\frac{1}{(n+2)(n+3)} = \frac{C_1}{n+2} + \frac{C_2}{n+3}$

   $\Rightarrow \frac{1}{(n+2)(n+3)}(n+2)(n+3) = \frac{C_1}{n+2}(n+2)(n+3) + \frac{C_2}{n_3}(n+2)(n+3)$

   $\Rightarrow 1 = C_1(n+3) + C_2(n+2)$

   $\boxed{n=-3}$ $\Rightarrow 1 = C_2(-1)$

   $\boxed{\Rightarrow C_2 = -1}$

$\boxed{n=-2}\quad \Rightarrow 1 = C_1\,(1)$

$\boxed{\Rightarrow C_1 = 1}$

Thus, $\frac{1}{n^2+5n+6} = \frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3}$

$$\Rightarrow \sum_{n=1}^{N} \frac{1}{n^2+5n+6} = \sum_{n=1}^{N} \left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \dots$$

$$+ \left(\frac{1}{N} - \frac{1}{N+1}\right) + \left(\frac{1}{N+1} - \frac{1}{N+2}\right) + \left(\frac{1}{N+2} - \frac{1}{N+3}\right)$$

$$= \sum_{n=1}^{N} \left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{1}{3} - \frac{1}{N+3}$$

i.e., $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6} = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \lim_{N\to\infty} \left(\frac{1}{3} - \frac{1}{N+3}\right) = \frac{1}{3}$

$$\boxed{\text{i.e., } \sum_{n=1}^{\infty} \frac{1}{n^2+5n+6} = \frac{1}{3}}$$

In Exercises 5-6, determine convergence/divergence of the given series. (Justify your answers!) **If the series converges, determine its sum.**

5. $1 + \frac{3}{5} + \frac{9}{25} + \frac{27}{125} + \dots + \left(\frac{3}{5}\right)^n + \dots$

   If the series converges, determine its sum. In general, there are only two types of convergent series whose sums we can compute: Geometric and "Telescoping Sum."

   Notice that each term after the first term is equal to $\frac{3}{5}$ times its predecessor.

   The series is geometric with ratio $r = \frac{3}{5}$

   Since $|r| < 1$, the series converges to $\frac{1^{\text{st}}\text{ term}}{1-r} = \frac{1}{1-\frac{3}{5}} = \frac{1}{\left(\frac{2}{5}\right)} = \frac{5}{2}$

   $\boxed{\text{The series \textbf{converges} to } \frac{5}{2}}$

6. $\sum_{n=1}^{\infty} \frac{n}{n+5} =$

There are a few different ways that we can do this.

First, note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+5} = 1$

Since $\lim_{n\to\infty} a_n \neq 0$, the series **diverges.**

i.e., $\sum_{n=1}^{\infty} \frac{n}{n+5}$ **diverges** by the "n^{th} term Test."

In Exercises 7-8, determine convergence/divergence of the given series. (Justify your answers!)

7. $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}-1}$

There are a few different ways that we can do this.

We can compare $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}-1}$ to $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}}$, which is a p-series with $p = \frac{1}{2}$.

Hence $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges.

Since $\underbrace{\dfrac{1}{n^{\frac{1}{2}}}}_{a_n} < \underbrace{\dfrac{1}{n^{\frac{1}{2}}-1}}_{b_n}$ and $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges, $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}-1}$ diverges also, by the **Direct Comparison Test.**

Alternatively: Observe that $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\left(\frac{1}{n^{\frac{1}{2}}}\right)}{\left(\frac{1}{n^{\frac{1}{2}}-1}\right)}\right| = \lim_{n\to\infty} \frac{n^{\frac{1}{2}}-1}{n^{\frac{1}{2}}} = 1$

Since $0 < \lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| < \infty$, Both series "do the same thing."

Since $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}}$, is a divergent p-series (with $p = \frac{1}{2}$), $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}-1}$ diverges also, by the **Limit Comparison Test.**

i.e., $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}-1}$ **diverges** by **Direct Comparison** and **Limit Comparison** with $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}}$

8. $\sum_{n=1}^{\infty} \frac{1}{n+3}$

There are a few ways to do this.

First, we can compare $\sum_{n=1}^{\infty} \frac{1}{n+3}$ with $\sum_{n=1}^{\infty} \frac{1}{n}$, which is the divergent Harmonic Series.

Since $\underbrace{\frac{1}{n+3}}_{a_n} < \underbrace{\frac{1}{n}}_{b_n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the Direct Comparison Test doesn't apply.

Applying the Limit Comparison Test, we have: $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\left(\frac{1}{n+3}\right)}{\left(\frac{1}{n}\right)}\right| = \lim_{n\to\infty} \frac{n}{n+3} = 1$

Since $0 < \lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| < \infty$, Both series "do the same thing."

Since $\sum_{n=1}^{\infty} \frac{1}{n}$, is the divergent Harmonic Series, $\sum_{n=4}^{\infty} \frac{1}{n+3}$ diverges also, by the **Limit Comparison Test.**

Alternatively, $\int_1^{\infty} \frac{1}{n+3} dn = \lim_{b\to\infty} \int_1^b \underbrace{\frac{1}{n+3}}_{\frac{1}{u}} \underbrace{dn}_{du} = \lim_{b\to\infty} \left[\ln(n+3)\right]_1^b =$

$\lim_{b\to\infty} \left[\ln(b+3) - \ln(1)\right] = \infty$

$\sum_{n=4}^{\infty} \frac{1}{n+3}$ **diverges** by the **Integral Test**

$\sum_{n=4}^{\infty} \frac{1}{n+3}$ **diverges** by the **Integral Test** and by **Limit Comparison** with $\sum_{n=4}^{\infty} \frac{1}{n}$

For exercises 9-10, choose one. (You can do the other for extra credit. (10 points))

9. Determine convergence/divergence of the given series. (Justify your answer!)

$$\sum_{n=1}^{\infty} \left(\frac{n+2}{2n+1}\right)^n$$

The n^{th} a_n is something **raised to the** n^{th} **power,** so this is a good candidate for the n^{th} **Root Test.**

Observe: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+2}{2n+1}\right)^n} = \lim_{n\to\infty} \left(\frac{n+2}{2n+1}\right) = \lim_{n\to\infty} \frac{n}{2n} = \frac{1}{2}$

Since $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, the series **converges.** by the n^{th} **Root Test.**

$\sum_{n=1}^{\infty} \left(\frac{n+2}{2n+1}\right)^n$ **converges** by the n^{th} **Root Test.**

10. Determine convergence/divergence of the given series. (Justify your answer!)

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

The n^{th} term a_n contains a **factorial,** so this is a good candidate for the **Ratio Test.**

Observe: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}}{(n+1)!}}{\left(\frac{2^n}{n!}\right)}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0$

Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series **converges.**

$\sum_{n=1}^{\infty} \frac{2^n}{n!}$ **converges** by the **Ratio Test.**

Extra Wow! (10 points)

Determine convergence/divergence of the given series. (Justify your answer!)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \tfrac{1}{\sqrt{n}} = 1 - \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{3}} - \tfrac{1}{2} + \dots$$

Observe: Our series fits the form: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{\frac{1}{\sqrt{n}}}_{a_n} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$

The terms of the series are alternately positive and negative.

Also: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \quad$ (i.e., $\lim_{n\to\infty} a_n = 0$)

And: $\displaystyle\underbrace{\frac{1}{\sqrt{n+1}}}_{a_{n+1}} \leq \underbrace{\frac{1}{\sqrt{n}}}_{a_n}$

By the Alternating Series Test, the series converges.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ **converges** by the **Alternating Series Test**