Proofs Involving Sets #7 (Proving Statements False by Counterexample) - Solutions

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Instructions. Disprove by providing a counter-example.

1. **Proof.** $(A \cup B)^c = A^c \cup B^c$

   Counter-example: $A = \{1, 2\}$; $B = \{2, 3\}$; $U = \{1, 2, 3\}$.

   Observe:

   $A \cup B = \{1, 2, 3\}$

   $(A \cup B)^c = \emptyset$

   $A^c = \{3\}$

   $B^c = \{1\}$

   $A^c \cup B^c = \{1, 3\}$

   $(A \cup B)^c = \emptyset \neq \{1, 3\} = A^c \cup B^c$

   i.e., $(A \cup B)^c \neq A^c \cup B^c$

   Hence, the statement $(A \cup B)^c = A^c \cup B^c$ is false by counter-example. ■

   Alternatively:

   Proof. Counter-example: $A = \emptyset$; $B = U$, where U is non-empty.

   Observe:

   $A \cup B = U$

   $(A \cup B)^c = \emptyset$

   $A^c = U$

   $B^c = \emptyset$

   $A^c \cup B^c = U$

   $(A \cup B)^c = \emptyset \neq U = A^c \cup B^c$

   i.e., $(A \cup B)^c \neq A^c \cup B^c$

   Hence, the statement $(A \cup B)^c = A^c \cup B^c$ is false by counter-example. ■

2. If A is a set with an infinite number of elements, then every subset of A has an infinite number of elements.

   Proof. Counter-example: Consider the set of natural numbers $\mathbf{N} = \{1, 2, 3, \ldots\}$.

   $\mathbf{N}$ is a set with an infinite number of elements, but $A = \{1, 2\}$ is a subset of $\mathbf{N}$ that has only finitely many elements.

   Hence the original statement is false by counter-example. ■

3. $A \subseteq (A \cap B)$

   Proof. Counter-example: $A = \{1, 2\}$; $B = \{2, 3\}$
   Observe: $(A \cap B) = \{2\}$
   $A = \{1, 2\} \nsubseteq \{2\} = (A \cap B)$

   i.e., $A \nsubseteq (A \cap B)$

   Hence, the statement $A \subseteq (A \cap B)$ is false by counter-example. ■

   Alternatively:

   Proof. Counter-example: $A = U$, where U is non-empty, and $B = \emptyset$.

   Observe:

   $A \cap B = \emptyset$

   $A = U \nsubseteq \emptyset = (A \cap B)$

   i.e., $A \nsubseteq (A \cap B)$

   Hence, the statement $A \subseteq (A \cap B)$ is false by counter-example. ■