MTH 3311 – Test #2 Part #1 – Solutions

Spring 2020

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Directions: Show CLEARLY how you arrive at your answers.

1. Hey everybody – a while ago, I took a beer and an empty beer mug out of the freezer. The beer and the mug were at 28 °F when they were removed from the freezer (just above freezing!) and I poured the beer into the mug and put it on a table. After 10 minutes, the temperature of the beer was 35 °F. If the temperature of the air in the room is maintained at a constant temperature of 70 °F,

   (a) derive an equation for the temperature of the beer at any time $t \geq t_0$ (where $t_0 = 0\,\mathrm{s}$ is the time when the beer is first removed from the freezer.)

   (b) At what time will the temperature of my beer be 40 °F ?

Let T = temperature of the beer

t = time

T_r = room temperature = 70 °F

According to **Newton's Law of Cooling,** the rate of change of beer temperature, with respect to time, is proportional to the **difference** between the air and beer temperatures.

Hence, $\frac{dT}{dt} = k\,(T - T_r)$ (Where k is the constant of proportionality)

$\Rightarrow \frac{dT}{(T-T_r)} = kdt$

$\Rightarrow \int \frac{1}{(T-T_r)} dT = \int kdt$

$\Rightarrow \ln|T - T_r| = kt + C$ (We can assume that $(T - T_r) < 0$, since $T < T_r$ initially. Hence, if we want to remove the absolute value bars, we have to acknowledge that, since $(T - T_r) < 0$, it follows that $|T - T_r| = -(T - T_r) = (T_r - T)$.)

Thus, we have: $\Rightarrow \ln|T - T_r| = kt + C$

$\Rightarrow \ln(T_r - T) = kt + C$

(Alternative Approach on the next page)

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Alternatively, since $T < T_r$ initially (i.e., $T_r > T$), we could take the equation, $\frac{dT}{dt} = k(T - T_r)$ and manipulate it as follows: $\frac{dT}{dt} = k(T - T_r) \Rightarrow \frac{dT}{dt} = -k(T_r - T)$, eliminating the need to worry about absolute value.

$\Rightarrow \frac{dT}{(T_r - T)} = -kdt$

$\Rightarrow \int \frac{1}{(T_r - T)} dT = \int -kdt$

$\Rightarrow -\ln|T_r - T| = -kt + C$

$\Rightarrow \ln|T_r - T| = kt + C$ (Since $(T_r - T) > 0$, it follows that $|T_r - T| = (T_r - T)$.)

Thus, we have: $\ln(T_r - T) = kt + C$

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Regardless of our choice of approach, we have: $\ln(T_r - T) = kt + C$

$\Rightarrow e^{\ln(T_r - T)} = e^{kt + C} = C_1 e^{kt}$

i.e., $T_r - T = C_1 e^{kt}$

(Since $T_r = 70\,°\mathrm{F}$ is constant, we put the value in here.)

i.e., $70\,°\mathrm{F} - T = C_1 e^{kt}$

$\Rightarrow 70\,°\mathrm{F} - C_1 e^{kt} = T$

$\Rightarrow T = 70\,°\mathrm{F} - C_1 e^{kt}$

Notice that we have two constants to evaluate.

Therefore, we need two initial conditions

Recall: at $t = 0\,\mathrm{min}$, $T = 28\,°\mathrm{F}$

$\Rightarrow 28\,°\mathrm{F} = T(0\,\mathrm{min}) = 70\,°\mathrm{F} - C_1 e^{k(0\,\mathrm{min})}$

$\Rightarrow 28\,°\mathrm{F} = 70\,°\mathrm{F} - C_1$

$\Rightarrow -42\,°\mathrm{F} = -C_1$

$\Rightarrow C_1 = 42\,°\mathrm{F}$

Thus, $T(t) = 70\,°\mathrm{F} - 42\,°\mathrm{F} e^{kt}$

Recall Also: at $t = 10\,\text{min}$, $T = 35\,°\text{F}$

This yields: $35\,°\text{F} = T\,(10\,\text{min}) = 70\,°\text{F} - 42\,°\text{F}e^{k(10\,\text{min})}$

i.e., $35\,°\text{F} = 70\,°\text{F} - 42\,°\text{F}e^{k(10\,\text{min})}$

$\Rightarrow -35\,°\text{F} = -42\,°\text{F}e^{k(10\,\text{min})}$

$\Rightarrow \frac{-35\,°\text{F}}{-42\,°\text{F}} = e^{k(10\,\text{min})}$

$\Rightarrow \ln\left(\frac{5}{6}\right) = \ln\left(e^{k(10\,\text{min})}\right)$

$\Rightarrow \ln\left(\frac{5}{6}\right) = k\,(10\,\text{min})$

$\Rightarrow \frac{\ln\left(\frac{5}{6}\right)}{(10\,\text{min})} = k$

$\Rightarrow k = \frac{\ln\left(\frac{5}{6}\right)}{10\,\text{min}}$

$$\Rightarrow T\,(t) = 70\,°\text{F} - 42\,°\text{F}e^{\frac{\ln\left(\frac{5}{6}\right)}{10\,\text{min}}t}$$ OR $$\Rightarrow T\,(t) = 70\,°\text{F} - 42\,°\text{F}e^{-\frac{0.0182}{\text{min}}t}$$

b) At what time will the temperature of my beer be 40 °F ?

$40\,°\text{F} = T\,(t) = 70\,°\text{F} - 42\,°\text{F}e^{\frac{\ln\left(\frac{5}{6}\right)}{10\,\text{min}}t}$

i.e., $40\,°\text{F} = 70\,°\text{F} - 42\,°\text{F}e^{\frac{\ln\left(\frac{5}{6}\right)}{10\,\text{min}}t}$

$\Rightarrow -30\,°\text{F} = -42\,°\text{F}e^{\frac{\ln\left(\frac{5}{6}\right)}{10\,\text{min}}t}$

$\Rightarrow \frac{-30\,°\text{F}}{-42\,°\text{F}} = e^{\frac{\ln\left(\frac{5}{6}\right)}{10\,\text{min}}t}$

$\Rightarrow \frac{5}{7} = e^{\frac{\ln\left(\frac{5}{6}\right)}{10\,\text{min}}t}$

$\Rightarrow \ln\left(\frac{5}{7}\right) = \ln\left(e^{\frac{\ln\left(\frac{5}{6}\right)}{10\,\text{min}}t}\right) = \frac{\ln\left(\frac{5}{6}\right)}{10\,\text{min}}t$

i.e. $\ln\left(\frac{5}{7}\right) = \frac{\ln\left(\frac{5}{6}\right)}{10\,\text{min}}t$

$\Rightarrow \frac{\ln\left(\frac{5}{7}\right)(10\,\text{min})}{\ln\left(\frac{5}{6}\right)} = t$

i.e., $t = \frac{\ln\left(\frac{5}{7}\right)(10\,\text{min})}{\ln\left(\frac{5}{6}\right)} = 18.45\,\text{min}$

At $t = 18.45\,\text{min}$, $T = 40\,°\text{F}$