Integrals and Natural Logarithms #1 - Solutions

Spring 2013

Pat Rossi Name ______________________________

Instructions

Answers appear on the ANSWERS page. Solutions appear on the SOLUTIONS page.

1. Compute: $\int \left(5x^4 + 4x^3 + 6x + 6\right) dx =$

   $\int \left(5x^4 + 4x^3 + 6x + 6\right) dx = 5\left[\frac{x^5}{5}\right] + 4\left[\frac{x^4}{4}\right] + 6\left[\frac{x^2}{2}\right] + 6x + C$

   i.e., $\int \left(5x^4 + 4x^3 + 6x + 6\right) dx = x^5 + x^4 + 3x^2 + 6x + C$
   Don't forget the "+C"

2. Compute: $\int (\sin(x) + \sec(x)\tan(x))\, dx =$

   $\int (\sin(x) + \sec(x)\tan(x))\, dx = [-\cos(x)] + [\sec(x)] + C$

   i.e., $\int (\sin(x) + \sec(x)\tan(x))\, dx = -\cos(x) + \sec(x) + C$
   Don't forget the "+C"

3. Compute: $\int_{x=1}^{x=2} \left(6x^3 + 4x^2 + 4x\right) dx =$

   $$\int_{x=1}^{x=2} \underbrace{\left(6x^3 + 4x^2 + 4x\right)}_{f(x)} dx = \underbrace{\left[\frac{3}{2}x^4 + \frac{4}{3}x^3 + 2x^2\right]_{x=1}^{x=2}}_{F(x)}$$

   $$= \underbrace{\left[\frac{3}{2}(2)^4 + \frac{4}{3}(2)^3 + 2(2)^2\right]}_{F(2)} - \underbrace{\left[\frac{3}{2}(1)^4 + \frac{4}{3}(1)^3 + 2(1)^2\right]}_{F(1)} = \frac{227}{6}$$

   i.e., $\int_{x=1}^{x=2} \left(6x^3 + 4x^2 + 4x\right) dx = \frac{227}{6}$

4. Compute: $\int \left(8x^3+12x^2\right)^{10}\left(x^2+x\right)dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(8x^3+12x^2\right)^{10}$ (A function raised to a power is always a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(8x^3+12x^2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(8x^3+12x^2\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(x^2+x\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(8x^3+12x^2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 8x^3+12x^2 \\ \Rightarrow \frac{du}{dx} &= 24x^2+24x \\ \Rightarrow du &= \left(24x^2+24x\right)dx \\ \Rightarrow \frac{1}{24}du &= \left(x^2+x\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(8x^3+12x^2\right)^{10}}_{u^{10}}\underbrace{\left(x^2+x\right)dx}_{\frac{1}{24}du} = \int u^{10}\frac{1}{24}du = \frac{1}{24}\int u^{10}du$$

4. Integrate (in terms of u).

$$\frac{1}{24}\int u^{10}du = \frac{1}{24}\left[\frac{u^{11}}{11}\right]+C = \frac{1}{264}u^{11}+C$$

5. Re-express in terms of the original variable, x.

$$\int \left(8x^3+12x^2\right)^{10}\left(x^2+x\right)dx = \underbrace{\frac{1}{264}\left(8x^3+12x^2\right)^{11}+C}_{\frac{1}{264}u^{11}+C}$$

i.e., $\int \left(8x^3+12x^2\right)^{10}\left(x^2+x\right)dx = \frac{1}{264}\left(8x^3+12x^2\right)^{11}+C$

5. Compute: $\int \sin\left(x^3+3x^2\right)\left(6x^2+12x\right)dx =$

   1. Is u-sub appropriate?

      a. Is there a composite function?

      Yes! $\sin\left(x^3+3x^2\right)$

      $\nearrow$ outer $\quad\quad$ $\nwarrow$ inner

      Let $u =$ the "inner" of the composite function

      $\Rightarrow u = \left(x^3+3x^2\right)$

      b. Is there an (approximate) function/derivative pair?

      Yes! $\underbrace{\left(x^3+3x^2\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(6x^2+12x\right)}_{\text{deriv}}$

      Let $u =$ the "function" of the function/deriv pair

      $\Rightarrow u = \left(x^3+3x^2\right)$

      c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

      (i.e., do criteria **a** and **b** suggest the same choice of u ?)

      Yes! $\Rightarrow$ u-substitution is appropriate

   2. Compute du

$$\begin{aligned} u &= x^3+3x^2 \\ \Rightarrow \frac{du}{dx} &= 3x^2+6x \\ \Rightarrow du &= \left(3x^2+6x\right)dx \\ \Rightarrow 2du &= \left(6x^2+12x\right)dx \end{aligned}$$

   3. Analyze in terms of u and du

$$\int \underbrace{\sin\left(x^3+3x^2\right)}_{\sin(u)}\underbrace{\left(6x^2+12x\right)dx}_{2du} = \int \sin(u)\ 2du = 2\int \sin(u)\,du$$

   4. Integrate (in terms of u).

$$2\int \sin(u)\,du = 2\left[-\cos(u)\right]+C = -2\cos(u)+C$$

   5. Re-express in terms of the original variable, x.

$$\int \sin\left(x^3+3x^2\right)\left(6x^2+12x\right)dx = \underbrace{-2\cos\left(x^3+3x^2\right)+C}_{-2\cos(u)+C}$$

i.e., $\int \sin\left(x^3+3x^2\right)\left(6x^2+12x\right)dx = -2\cos\left(x^3+3x^2\right)+C$

6. Compute: $\int \frac{x+1}{3x^2+6x}dx =$

$$\int \frac{x+1}{3x^2+6x}dx \underbrace{=}_{\text{re-write}} \int \frac{1}{3x^2+6x}(x+1)\,dx$$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{3x^2+6x}$ is the same as $\left(3x^2+6x\right)^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(3x^2+6x\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(3x^2+6x\right)}_{\text{function}} - - - - \rightarrow \underbrace{(x+1)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(3x^2+6x\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 3x^2+6x \\ \Rightarrow \frac{du}{dx} &= 6x+6 \\ \Rightarrow du &= (6x+6)\,dx \\ \Rightarrow \frac{1}{6}du &= (x+1)\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\frac{1}{3x^2+6x}}_{\frac{1}{u}} \underbrace{(x+1)\,dx}_{\frac{1}{6}du} = \int \frac{1}{u}\cdot\frac{1}{6}du = \frac{1}{6}\int \frac{1}{u}du$$

4. Integrate (in terms of u).

$$\frac{1}{6}\int \frac{1}{u}du = \frac{1}{6}\left[\ln|u|\right] + C = \frac{1}{6}\ln|u| + C$$

5. Re-express in terms of the original variable, x.

$$\int \frac{x+1}{3x^2+6x}dx = \underbrace{\frac{1}{6}\ln\left|3x^2+6x\right| + C}_{\frac{1}{6}\ln|u|+C}$$

i.e., $\int \frac{x+1}{3x^2+6x}dx = \frac{1}{6}\ln\left|3x^2+6x\right| + C$

7. Compute: $\frac{d}{dx}\left[\ln\left(\sin\left(x\right)\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(\sin\left(x\right)\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{\sin\left(x\right)}}_{\frac{1}{g(x)}} \cdot \underbrace{\cos\left(x\right)}_{g'(x)} = \frac{\cos(x)}{\sin(x)} = \cot\left(x\right)$$

i.e., $\frac{d}{dx}\left[\ln\left(\sin\left(x\right)\right)\right] = \frac{\cos(x)}{\sin(x)} = \cot\left(x\right)$

8. Compute: $\frac{d}{dx}\left[\ln\left(3x^3 - 9x + 5\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(3x^3 - 9x + 5\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{3x^3 - 9x + 5}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(9x^2 - 9\right)}_{g'(x)} = \frac{9x^2-9}{3x^3-9x+5}$$

i.e., $\frac{d}{dx}\left[\ln\left(3x^3 - 9x + 5\right)\right] = \frac{9x^2-9}{3x^3-9x+5}$

9. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2-1}{x}}\right)\right] \underbrace{=}_{\text{re-write}} \frac{d}{dx}\left[\ln\left[\left(\frac{x^2-1}{x}\right)^{\frac{1}{2}}\right]\right]$

Remark: We can compute this derivative directly, in it's current form, but it would be much easier to use the *algebraic properties* of natural logarithms to simplify the expression first.

$$\frac{d}{dx}\left[\ln\left[\left(\frac{x^2-1}{x}\right)^{\frac{1}{2}}\right]\right] = \underbrace{\frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{x^2-1}{x}\right)\right]}_{\ln(a^n) = n\ln(a)} = \underbrace{\frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(x^2-1\right) - \ln\left(x\right)\right)\right]}_{\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)} = \frac{1}{2}\frac{d}{dx}\left[\ln\left(x^2-1\right) - \ln\left(x\right)\right]$$

NOW we're ready to compute the derivative!

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2-1}{x}}\right)\right] = \frac{1}{2}\frac{d}{dx}\left[\ln\left(x^2-1\right) - \ln\left(x\right)\right] = \frac{1}{2}\left[\frac{1}{x^2-1}\left(2x\right) - \frac{1}{x}\right] = \frac{x}{x^2-1} - \frac{1}{2x}$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2-1}{x}}\right)\right] = \frac{x}{x^2-1} - \frac{1}{2x}$

10. Compute: $\int_{x=-1}^{x=1} \left(x^2 - 3x + 1\right)^3 (8x - 12)\, dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(x^2 - 3x + 1\right)^3$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(x^2 - 3x + 1\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(x^2 - 3x + 1\right)}_{\text{function}} - - - - \rightarrow \underbrace{(8x - 12)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(x^2 - 3x + 1\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= x^2 - 3x + 1 \\ \Rightarrow \frac{du}{dx} &= 2x - 3 \\ \Rightarrow du &= (2x - 3)\, dx \\ \Rightarrow 4du &= (8x - 12)\, dx \end{aligned}$$

   When $x = -1$, $u = x^2 - 3x + 1 = (-1)^2 - 3(-1) + 1 = 5$
   When $x = 1$, $u = x^2 - 3x + 1 = (1)^2 - 3(1) + 1 = -1$

3. Analyze in terms of u and du

$$\int_{x=-1}^{x=1} \underbrace{\left(x^2 - 3x + 1\right)^3}_{u^3} \underbrace{(8x - 12)\, dx}_{4du} = \int_{u=5}^{u=-1} u^3 \cdot 4du = 4\int_{u=5}^{u=-1} u^3 du$$

   Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$4\int_{u=5}^{u=-1} u^3 du = 4\left[\frac{u^4}{4}\right]_{u=5}^{u=-1} = \left[u^4\right]_{u=5}^{u=-1} = \underbrace{(-1)^4}_{F(-1)} - \underbrace{(5)^4}_{F(5)} = -624$$

$$\text{i.e., } \int_{x=-1}^{x=1} \left(x^2 - 3x + 1\right)^3 (8x - 12)\, dx = -624$$