MTH 1126 - Test #3 - 9am Class - Solutions

Spring 2024

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Show CLEARLY how you arrive at your answers.

1. $\int \frac{2x-22}{x^2-x-12} dx =$

Note that $\int \frac{2x-22}{x^2-x-12} dx$ does not fit the form: $\int \frac{1}{u} du$

Therefore, we decompose $\frac{2x-22}{x^2-x-12}$ into the sum of simpler quotients:

1. Make sure that deg(numerator) $\leq$ deg(denominator)

2. Factor the denominator.

$\frac{2x-22}{x^2-x-12} = \frac{2x-22}{(x+3)(x-4)}$

3. For each linear factor $(x+c)$, form the term $\frac{C_1}{x+c}$

$\frac{2x-22}{x^2-x-12} = \frac{2x-22}{(x+3)(x-4)} = \frac{C_1}{x+3} + \frac{C_2}{x-4}$

4. Solve for the constants

$\frac{2x-22}{(x+3)(x-4)} = \frac{C_1}{x+3} + \frac{C_2}{x-4}$

$\Rightarrow \frac{2x-22}{(x+3)(x-4)} (x+3)(x-4) = \frac{C_1}{x+3} (x+3)(x-4) + \frac{C_2}{x-4} (x+3)(x-4)$

i.e., $2x - 22 = C_1 (x-4) + C_2 (x+3)$

Plug in "strategic values" of x to find the values of the constants.

$\boxed{x = -3}$

$\Rightarrow -28 = -7C_1$

$\Rightarrow \boxed{C_1 = 4}$

$\boxed{x = 4}$

$\Rightarrow -14 = 7C_2$

$\Rightarrow \boxed{C_2 = -2}$

Thus, $\frac{2x-22}{(x+3)(x-4)} = \frac{C_1}{x+3} + \frac{C_2}{x-4} = \frac{4}{x+3} - \frac{2}{x-4}$

i.e., $\frac{2x-22}{(x+3)(x-4)} = \frac{4}{x+3} - \frac{2}{x-4}$

Consequently, $\int \frac{2x-22}{x^2-x-12}dx = \int \left(\frac{4}{x+3} - \frac{2}{x-4}\right) dx = 4\int \frac{1}{x+3}dx - 2\int \frac{1}{x-4}dx$

$$= 4\ln|x+3| - 2\ln|x-4| + C$$

$$\boxed{\int \frac{2x-22}{x^2-x-12}dx = 4\ln|x+3| - 2\ln|x-4| + C}$$

2. $\lim_{x\to 0} \frac{\sin(x)}{x} =$

   $\lim_{x\to 0} \frac{\sin(x)}{x} \sim \frac{0}{0}$ (Use L'Hopital's Rule)

   $\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}[\sin(x)]}{\frac{d}{dx}[x]} = \lim_{x\to 0} \frac{\cos(x)}{1} = \frac{1}{1} = 1$

   $$\boxed{\lim_{x\to 0} \frac{\sin(x)}{x} = 1}$$

3. $\int \frac{3x^2+4x+27}{x^3+9x} dx =$

Note that $\int \frac{3x^2+4x+27}{x^3+9x} dx$ does not fit the form: $\int \frac{1}{u} du$

Therefore, we decompose $\frac{3x^2+4x+27}{x^3+9x}$ into the sum of simpler quotients:

1. Make sure that deg(numerator) $\leq$ deg(denominator)

2. Factor the denominator.

$\frac{3x^2+4x+27}{x^3+9x} = \frac{3x^2+4x+27}{x(x^2+9)}$ (x^2+9 is an "irreducible quadratic")

To see this, note that $x^2 \geq 0$. Therefore $x^2+9 \geq 9$. And consequently, $x^2+9 \neq 0$ for any value of x. This means that x^2+9 cannot be factored.

Alternatively, note that if we plug the coefficients of x^2+9 into the quadratic formula, we get: $x = \frac{-0 \pm \sqrt{0^2-4(1)(9)}}{2(1)} = \frac{-0 \pm \sqrt{-36}}{2}$.

The fact that we get a negative under the radical tells us that x^2+9 is irreducible.

3. For each linear factor $(x+c)$, form the term $\frac{C_1}{x+c}$.

3.a. For each irreducible quadratic ax^2+bx+c, form the term $\frac{Ax+B}{ax^2+bx+c}$

$\frac{3x^2+4x+27}{x^3+9x} = \frac{3x^2+4x+27}{x(x^2+9)} = \frac{C}{x} + \frac{Ax+B}{x^2+9}$

4. Solve for the constants

$\frac{3x^2+4x+27}{x(x^2+9)} = \frac{C}{x} + \frac{Ax+B}{x^2+9}$

$\Rightarrow \frac{3x^2+4x+27}{x(x^2+9)} x\left(x^2+9\right) = \frac{C}{x} x\left(x^2+9\right) + \frac{Ax+B}{x^2+9} x\left(x^2+9\right) = C\left(x^2+9\right) + (Ax+B)\,x$

i.e., $3x^2+4x+27 = C\left(x^2+9\right) + (Ax+B)\,x$

Plug in "strategic values" of x to find the values of the constants.

$\boxed{x=0}$

$\Rightarrow 27 = 9C$

$\Rightarrow \boxed{C=3}$

This yields: $3x^2+4x+27 = 3\left(x^2+9\right) + (Ax+B)\,x$

Simplifying both sides we have:

$3x^2+4x+27 = (A+3)\,x^2 + Bx + 27$

Comparing coefficients of x^2 on both sides, we see that:

$3 = A + 3$

$\Rightarrow \boxed{A = 0}$

Comparing coefficients of x on both sides, we see that:

$\boxed{B = 4}$

Thus, $\frac{3x^2+4x+27}{x(x^2+9)} = \frac{3}{x} + \frac{0x+4}{x^2+9} = \frac{3}{x} + \frac{4}{x^2+9}$

i.e., $\frac{3x^2+4x+27}{x^3+9x} = \frac{3x^2+2x+12}{x(x^2+4)} = \frac{3}{x} + \frac{4}{x^2+9}$

Consequently, $\int \frac{3x^2+4x+27}{x^3+9x}dx = \int \left(\frac{3}{x} + \frac{4}{x^2+9}\right) dx = 3\int \frac{1}{x}dx + 4\int \frac{1}{x^2+9}dx$

$$= 3\ln|x| + 4\left(\frac{1}{3}\arctan\left(\frac{x}{3}\right)\right) + C = 3\ln|x| + \frac{4}{3}\arctan\left(\frac{x}{3}\right) + C$$

$$\boxed{\int \frac{3x^2+4x+27}{x^3+9x}dx = 3\ln|x| + \frac{4}{3}\arctan\left(\frac{x}{3}\right) + C}$$

4. $\int \sin^4(x)\cos^3(x)\,dx =$

We have an odd power of $\cos(x)$.

1. Reserve a factor of $\cos(x)$ to serve as our "future du."

$= \int \sin^4(x)\cos^2(x)\underbrace{\cos(x)\,dx}_{\text{"future du"}}$

This means that we intend to let $u = \sin(x)$

2. Convert remaining cosines into sines

$= \int \sin^4(x)\left(\cos^2(x)\right)\underbrace{\cos(x)\,dx}_{\text{"future du"}}$

$= \int \sin^4(x)\left(1 - \sin^2(x)\right)\cos(x)\,dx$

$= \int \left(\sin^4(x) - \sin^6(x)\right)\cos(x)\,dx$

$$\boxed{\begin{array}{ll} \text{Let} & u = \sin(x) \\ \Rightarrow & \frac{du}{dx} = \cos(x) \\ \Rightarrow & du = \cos(x)\,dx \end{array}}$$

$= \int \left(u^4 - u^6\right)du$

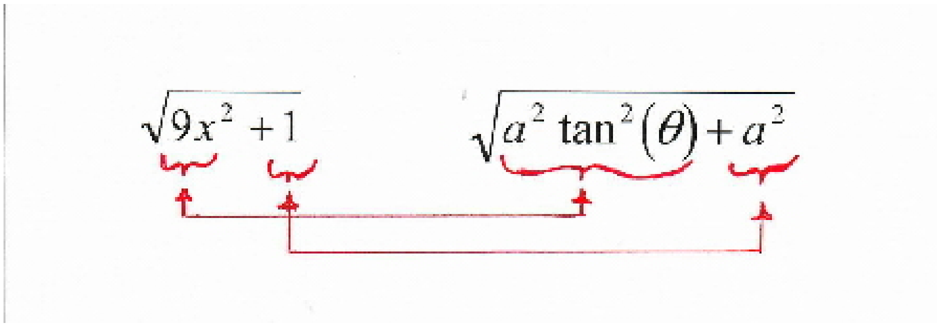
$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + C$

$= \frac{1}{5}\sin^5(x) - \frac{1}{7}\sin^7(x) + C$

$$\boxed{\int \sin^4(x)\cos^3(x)\,dx = \frac{1}{5}\sin^5(x) - \frac{1}{7}\sin^7(x) + C}$$

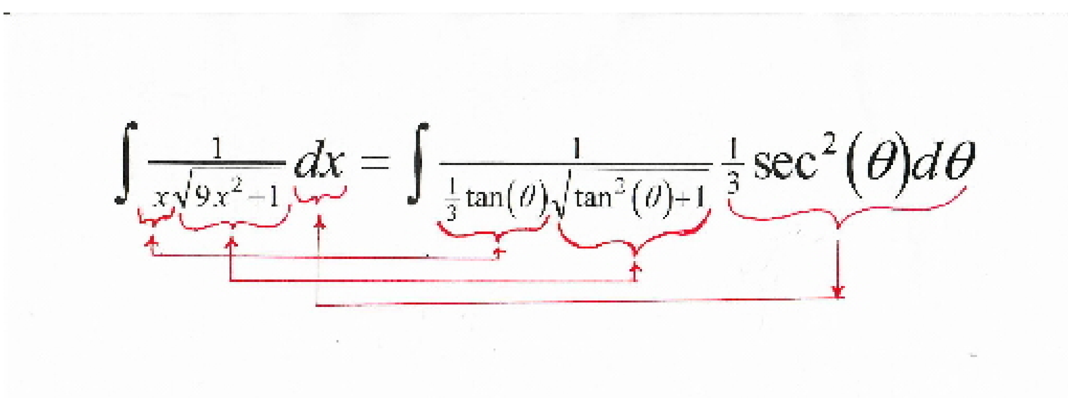
5. $\int \frac{1}{x\sqrt{9x^2+1}}dx =$

We match the radical $\sqrt{9x^2+1}$ with the radical $\sqrt{a^2\tan^2(\theta)+a^2}$



$$\boxed{\begin{array}{lrcl} & a^2 & = & 1 \\ \Rightarrow & a & = & 1 \\ & 9x^2 & = & a^2\tan^2(\theta) \\ \text{i.e.} & 9x^2 & = & \tan^2(\theta) \\ \Rightarrow & 3x & = & \tan(\theta) \\ \Rightarrow & x & = & \frac{1}{3}\tan(\theta) \\ \Rightarrow & \frac{dx}{d\theta} & = & \frac{1}{3}\sec^2(\theta) \\ \Rightarrow & dx & = & \frac{1}{3}\sec^2(\theta)\,d\theta \end{array}}$$

Rewrite the integral in terms of θ

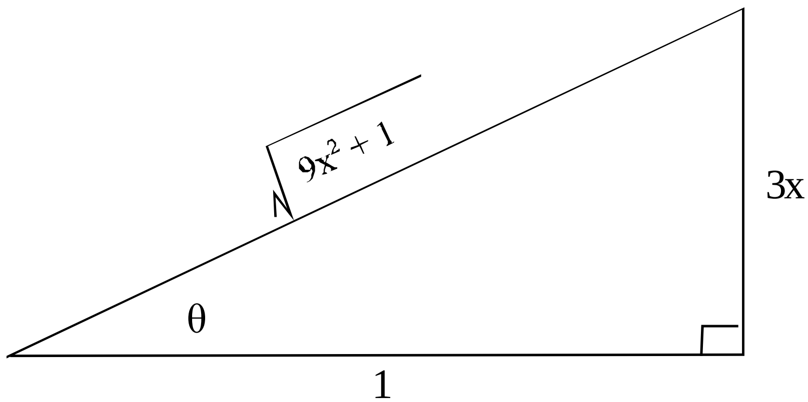


$$\int \frac{1}{x\sqrt{9x^2+1}}dx = \int \frac{1}{\frac{1}{3}\tan(\theta)\sqrt{\tan^2(\theta)+1}}\frac{1}{3}\sec^2(\theta)\,d\theta = \int \frac{1}{\frac{1}{3}\tan(\theta)\sqrt{\sec^2(\theta)}}\frac{1}{3}\sec^2(\theta)\,d\theta$$

$$= \int \frac{1}{\frac{1}{3}\tan(\theta)\sec(\theta)}\frac{1}{3}\sec^2(\theta)\,d\theta = \int \frac{1}{\tan(\theta)}\sec(\theta)\,d\theta = \int \cot(\theta)\sec(\theta)\,d\theta$$

$$= \int \frac{\cos(\theta)}{\sin(\theta)}\frac{1}{\cos(\theta)}d\theta = \int \frac{1}{\sin(\theta)}d\theta = \int \csc(\theta)\,d\theta = \ln|\csc(\theta) - \cot(\theta)| + C$$

To convert back to x, recall that $x = \frac{1}{3}\tan(\theta)$ (i.e., $\tan(\theta) = \frac{3x}{1} = \frac{\text{opp}}{\text{adj}}$)



$$\int \frac{1}{x\sqrt{9x^2+1}}dx = \ldots = \ln\left|\csc(\theta) - \cot(\theta)\right| + C = \ln\left|\frac{\sqrt{9x^2+1}}{3x} - \frac{1}{3x}\right| + C$$

$$\boxed{\int \frac{1}{x\sqrt{9x^2+1}}dx = \ln\left|\frac{\sqrt{9x^2+1}}{3x} - \frac{1}{3x}\right| + C}$$

6. $\lim_{x\to 0^+} x^2 \ln(x) =$

$\lim_{x\to 0^+} x^2 \ln(x) \; \tilde{} \; 0 \cdot (-\infty)$

In order to use L'Hôpital's Rule, we must rewrite this expression so that it is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$\lim_{x\to 0^+} x^2 \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-2}} \sim \frac{-\infty}{\infty}$ We can use L'Hôpital's Rule

$\lim_{x\to 0^+} x^2 \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-2}} = \lim_{x\to 0^+} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[x^{-2}]} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-2x^{-3}} = \lim_{x\to 0^+} \frac{1}{x} \cdot \left(-\frac{1}{2}\right) x^3$

$$= \lim_{x\to 0^+} -\frac{1}{2}x^2 = 0$$

i.e., $\lim_{x\to 0^+} x^2 \ln(x) = 0$

7. $\int x \sin(2x)\, dx =$

Use Integration by Parts

Our "Rules of Thumb" apply

Let u be the portion of the integrand whose derivative is simpler that itself.

$\Rightarrow u = x$

Let dv be the most complicated portion of the integrand that can be integrated

$\Rightarrow dv = \sin(2x)$

	$u = x$		$dv = \sin(2x)\, dx$
$\Rightarrow$	$\frac{du}{dx} = 1$	$\Rightarrow$	$v = \int \sin(2x)\, dx$
$\Rightarrow$	$du = dx$	$\Rightarrow$	$v = -\frac{1}{2}\cos(2x)$

$\Rightarrow \int x\sin(2x)\, dx = \int u dv = uv - \int v du = x\left(-\frac{1}{2}\cos(2x)\right) - \int \left(-\frac{1}{2}\cos(2x)\right) dx$

$$= -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x) = -\frac{1}{2}x\cos(2x) + \frac{1}{2}\left[\frac{1}{2}\sin(2x)\right] + C$$

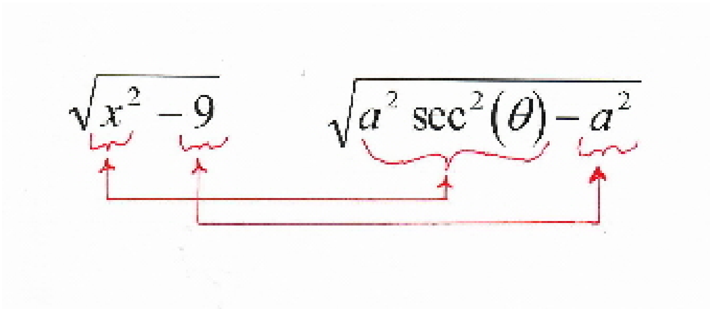
$$= -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C$$

$\int x\sin(2x)\, dx = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C$

WOW! Extra (10 pts - all or nothing)

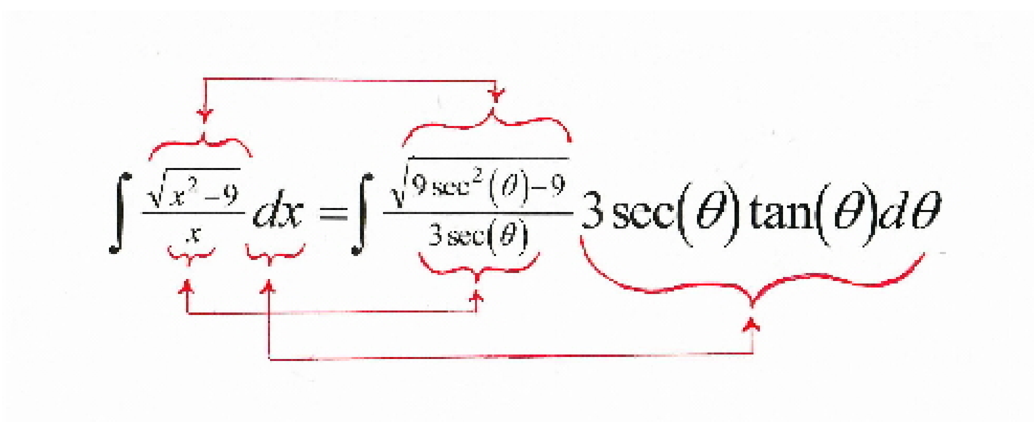
Compute: $\int \frac{\sqrt{x^2-9}}{x} dx =$

We match the radical $\sqrt{x^2-9}$ with the radical $\sqrt{a^2\sec^2(\theta)-a^2}$



$$\begin{array}{llcl}
 & a^2 & = & 9 \\
\Rightarrow & a & = & 3 \\
 & x^2 & = & a^2\sec^2(\theta) \\
\text{i.e.} & x^2 & = & 9\sec^2(\theta) \\
\Rightarrow & x & = & 3\sec(\theta) \\
\Rightarrow & \frac{d}{dx}[x] & = & \frac{d}{dx}[3\sec(\theta)] \\
\Rightarrow & 1 & = & 3\sec(\theta)\tan(\theta)\frac{d\theta}{dx} \\
\Rightarrow & dx & = & 3\sec(\theta)\tan(\theta)\,d\theta
\end{array}$$

Rewrite the integral in terms of θ



$$\int \frac{\sqrt{x^2-9}}{x} dx = \int \frac{\sqrt{9\sec^2(\theta)-9}}{3\sec(\theta)} 3\sec(\theta)\tan(\theta)\,d\theta = \int \frac{\sqrt{9\tan^2(\theta)}}{1}\tan(\theta)\,d\theta$$

$$= \int 3\tan(\theta)\cdot\tan(\theta)\,d\theta = 3\int \tan^2(\theta)\,d\theta = 3\int\left(\sec^2(\theta)-1\right)d\theta$$

$$= 3\left(\tan(\theta)-\theta\right)+C$$

Now we must re-express the answer in terms of the original variable x.

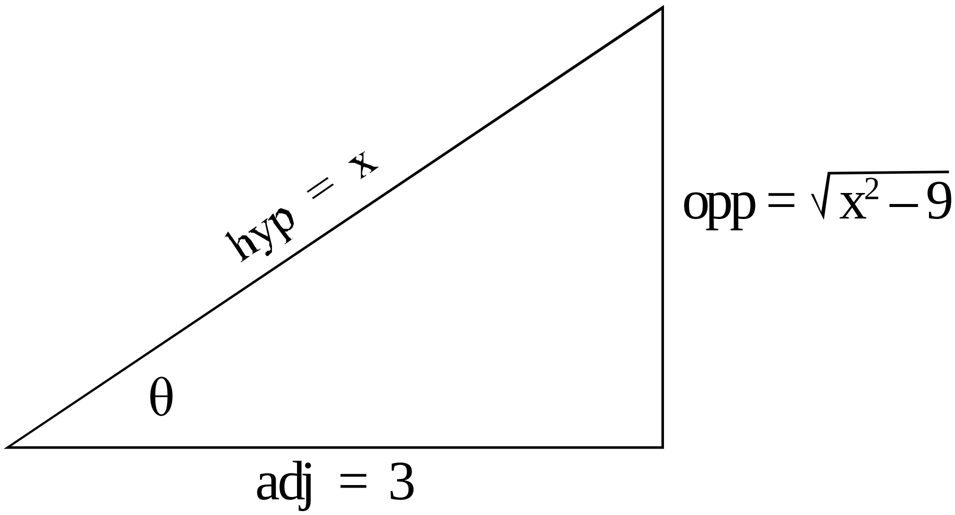
(Continued)

Recall: $x = 3\sec(\theta)$

$\Rightarrow \frac{x}{3} = \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\left(\frac{\text{adj}}{\text{hyp}}\right)} = \frac{\text{hyp}}{\text{adj}}$

i.e., $\frac{x}{3} = \frac{\text{hyp}}{\text{adj}}$

We will draw a right triangle that depicts this relationship.



From the diagram above, $\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x^2-9}}{3}$

i.e., $\tan(\theta) = \frac{\sqrt{x^2-9}}{3}$

To re-express θ in terms of x, we consider the equation $x = 3\sec(\theta)$

$\Rightarrow \frac{x}{3} = \sec(\theta)$

$\Rightarrow \sec^{-1}\left(\frac{x}{3}\right) = \sec^{-1}(\sec(\theta))$

$\Rightarrow \sec^{-1}\left(\frac{x}{3}\right) = \theta$

i.e., $\theta = \sec^{-1}\left(\frac{x}{3}\right)$

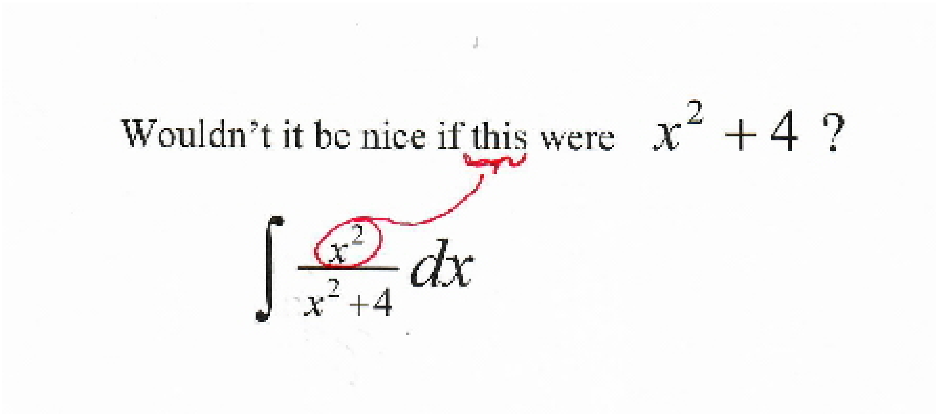
Thus, $\int \frac{\sqrt{x^2-9}}{x}dx = \ldots = 3(\tan(\theta) - \theta) + C = 3\left(\frac{\sqrt{x^2-9}}{3} - \sec^{-1}\left(\frac{x}{3}\right)\right) + C$

i.e., $\int \frac{\sqrt{x^2-9}}{x}dx = 3\left(\frac{\sqrt{x^2-9}}{3} - \sec^{-1}\left(\frac{x}{3}\right)\right) + C = \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right) + C$

$$\boxed{\int \frac{\sqrt{x^2-9}}{x}dx = 3\left(\frac{\sqrt{x^2-9}}{3} - \sec^{-1}\left(\frac{x}{3}\right)\right) + C = \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right) + C}$$

WOW! Extra (5 pts - all or nothing)

Compute: $\int \frac{x^2}{x^2+4} dx =$



$\int \frac{x^2}{x^2+4} dx = \int \left(\frac{x^2+4}{x^2+4} - \frac{4}{x^2+4} \right) dx = \int \left(1 - \frac{4}{x^2+4}\right) dx = \int 1 dx - \int \frac{4}{x^2+4} dx$

$= \int 1 dx - 4 \int \frac{1}{x^2+2^2} dx = x - 4 \left(\frac{1}{2} \arctan\left(\frac{x}{2}\right)\right) + C = x - 2 \arctan\left(\frac{x}{2}\right) + C$

$\boxed{\int \frac{x^2}{x^2+4} dx = x - 2 \arctan\left(\frac{x}{2}\right) + C}$